GRAPH COMPLEXES

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Abstract. This is a note from T. Willwacher's lecture series which was part of masterclass "Algebraic structure of Hochschild complexes" at the University of Copenhagen in October 2015.

From the course description: Graph complexes are differential graded vector spaces whose elements are linear combinations of combinatorial graphs. The differential is the operation of contracting an edge. These graph complexes exist in a variety of flavors (ribbon graphs, directed/undirected graphs etc.), each of which plays a central role in otherwise quite disjoint areas of mathematics like knot theory, geometric group theory and moduli spaces of curves. Despite the very elementary definition and its fundamental importance we know surprisingly little about what the graph homology actually is. The purpose of the course is to give an introduction to the problem and its origins, along with an overview of recent results.

Contents

1. General overview

1.1. Cochain complexes.

Definition 1.1. In this lecture we mainly work with cochain complexes: A $\mathbb{Z}$-*graded vector space* is defined as a vector space V, endowed with a direct sum decomposition

$$V = \bigoplus_{i\in\mathbb{Z}} V_i.$$

We call $v \in V_i$ a *vector of degree* i, denoted by $|v| = i$.

Suppose $V = \oplus V_i$ and $W = \oplus W_i$ are $\mathbb{Z}$-graded vector spaces, and $F\colon V \to W$ is a linear map. We say that F of *degree* n if

$$F(V_i) \subset W_{i+n}$$

holds for any i.

A *cochain complex*, or a *differential graded vector space*, is defined to be a pair (V, d), consisting of a $\mathbb{Z}$-graded vector space V and a linear map

$$d\colon V \to V,$$

called the *differential*, of degree 1 such that $d^2 = 0$ (this can be also expressed as $\operatorname{Ker} d \subset \operatorname{Im} d$). A *chain complex* is the same, but with d of degree -1 instead.

When (V, d) is a cochain complex, we call $v \in \operatorname{Ker} d$ *closed* and $v \in \operatorname{Im} d$ *exact*. We put $\mathrm{H}(V) = \operatorname{Ker} d / \operatorname{Im} d$, and call it the *cohomology* of (V, d). This is a $\mathbb{Z}$-graded vector space, where the grading is given by

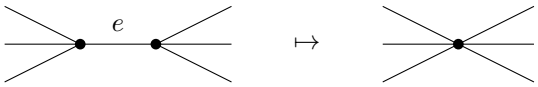
$$\mathrm{H}^i(V) = \operatorname{Ker}(d\colon V_i \to V_{i+1}) \Big/ \operatorname{Im}(d\colon V_{i-1} \to V_i)\,.$$

1.2. **Metadefinition of graph complexes.** There are many types of graph complexes, but all of them have the following form:

- Elements of graph complexes: formal linear combinations of isomorphism classes of (some type of) graphs
- A prescription to assign degrees to graphs
- the cochain differential is given by (alternating) sum of edge contractions:

$$d\Gamma = \sum_{e\colon \text{edges}} \pm \Gamma/e.$$

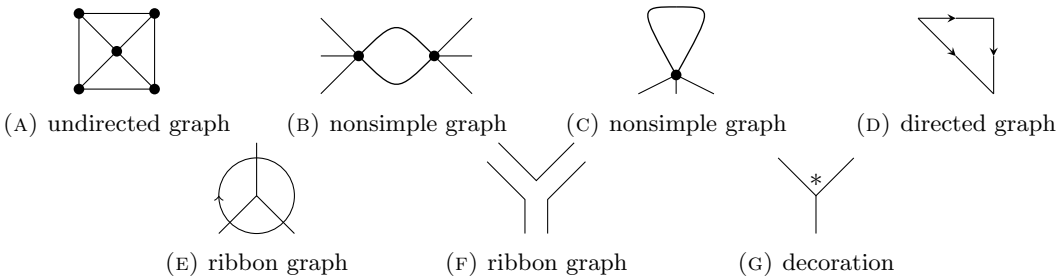
By an edge contraction, we mean an operation of the form



The signs are chosen in such a way that $d^2 = 0$ holds.

Given such a graph complex, we can consider its cohomology H(graph complex), which is called the *graph cohomology*. Examples of graph types are:

(1) Trees, planar trees (which have uninteresting cohomology).
(2) Undirected combinatorial graphs, like Figure 1A. We could add extra conditions on the graphs, such as:
   - the *valency* of edges being greater than 2,
   - being *connected*,
   - being *simple*, i.e., not allowing parts like Figures 1B and 1C.
   - being 1-*vertex irreducible*, i.e., the graph remains connected after removing any one vertex and its adjacent edges.
(3) Graphs with directed edges, like Figure 1D. Again with optional conditions such as:
   - being *acyclic* (no closed loops),
   - *ribbon graphs* (with a fixed cyclic ordering of edges at vertices), we can draw it like Figures 1E and 1F,
   - having extra decorations on edges or vertices, like Figure 1G.



(A) undirected graph (B) nonsimple graph (C) nonsimple graph (D) directed graph

(E) ribbon graph (F) ribbon graph (G) decoration

FIGURE 1. graphs

Remark 1.2. Apart from trees, graph cohomology in general is unknown!

1.3. **Classical problems reducible to graph cohomological computations.** Many interesting problems in mathematics can be recast in terms of graph cohomology computations. Let us list several such problems below.

1.3.1. *Cohomology of the moduli spaces of curves.* Consider the moduli space $\mathcal{M}_{g,n}$ of complex structures on an orientable surface $\Sigma_{g,n}$ of genus g with n labeled punctures.

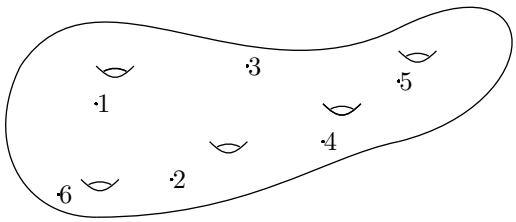


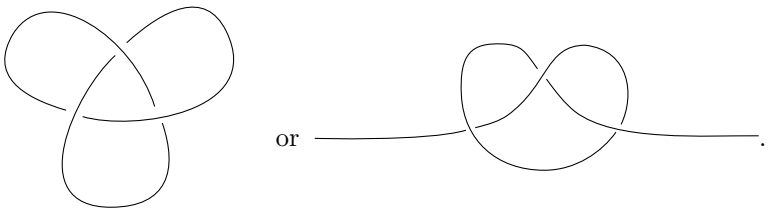
FIGURE 2. The surface $\Sigma_{5,6}$

It is an open problem to compute the cohomology of the moduli space, $\oplus_{n,g}\mathrm{H}(\mathcal{M}_{g,n},\mathbb{Q})$. One can also consider the case with unorderd punctures by considering the direct summand $\mathrm{H}(\mathcal{M}_{n,g})/S_n$, where the symmetric group S_n acts by permuting the labels of punctures. One can also make the punctures "odd" by tensoring with the sign representation to get

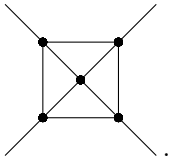
$$\bigoplus_{n,g} \left(\mathrm{H}(\mathcal{M}_{n,g}) \otimes \mathrm{sgn}_n\right)/_{S_n}.$$

In the stable setting and for $n \geq 1$, the computation of the $\mathrm{H}(\mathcal{M}_{g,n})$ can be recast as the computation of a ribbon graph complex (Penner [Pen88]). An extra differential from action on Lie bialgebras was found in joint work with Merkulov [MW15] and in another (yet unpublished) work of Arinkin and Caldararu. We will discuss this briefly in Section 5.

1.3.2. *Knot theory.* The goal of knot theory is to study embedding of S^1 (or $\mathbb{R}$ or $\mathbb{R}^m$) in $\mathbb{R}^3$ (or $\mathbb{R}^n$), so things like



The "finite type invariants" can be recast in terms of "hairy graphs", such as



When all vertices are trivalent, this gives rise to the Vassiliev invariants (these are conjecturally complete). Other classes give elements in $\pi_k(\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n))$. We will discuss about this in Section 3 more in detail.

1.3.3. *Outer automorphism groups of free groups.* Denote by $\mathbb{F}_n$ the *free group* on n generators $x_1, \dots, x_n$, i.e

$$\mathbb{F}_n = \{\text{words in } x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}\}/\sim,$$

where the equivalence relation $\sim$ is generated by $x_i x_i^{-1} = e$ for the neutral element e corresponding to the empty word. We denote by $\mathrm{Aut}(\mathbb{F}_n)$ the automorphisms of the group $\mathbb{F}_n$, and regard $\mathbb{F}_n$ as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$ realized as the inner automorphisms, i.e., $\phi_m(x) = mxm^{-1}$ for all $m, x \in \mathbb{F}_n$. Since the inner automorphisms form a normal subgroup $\mathrm{Aut}(\mathbb{F}_n)$, we can consider the quotient group

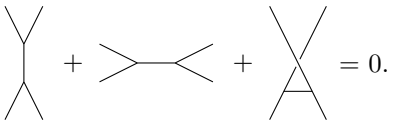
$$\mathrm{Out}(\mathbb{F}_n) := \mathrm{Aut}(\mathbb{F}_n)/\mathbb{F}_n$$

called the outer automorphisms. It is an open problem to compute the group cohomologies $\mathrm{H}^\bullet(\mathrm{Out}(\mathbb{F}_n))$ and $\mathrm{H}^\bullet(\mathrm{Aut}(\mathbb{F}_n))$. This problem can be recast as a graph cohomology computation. The relevant graph complex (see for example [Vog06]) is given by graphs with marked edges that cut the graph into trees, like Figure 3A. Another equivalent (and the more standard ap-

(A) graph with marked edges (B) graph with tree decoration

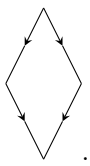
FIGURE 3. Presenting $\mathrm{H}^\bullet(\mathrm{Out}(\mathbb{F}_n))$

proach) way is to consider the graphs with vertices decorated by trivalent trees like Figure 3B, with the relation



1.3.4. *Deformation quantization.* This is "governed" by graph complexes of simple graphs [Dol11, Wil14], through the action of the graph operad on polyvector fields over $\mathbb{R}^d$.

1.3.5. *Quantum groups/Lie bialgebras.* Universal deformations of Lie bialgebra structures (equivalent classes of quantizations) are governed [Wil15a] by directed acyclic graph complexes, like



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Classes of degree 1 give universal deformations.

1.3.6. *Diffeomorphism groups of spheres.* Graph complexes give special classes in $\mathrm{H}^*(\mathrm{Diff}(S^n))$.

1.3.7. *Deformation theory of* E_n *Operads.* These are governed by "simple" graph complexes, in the sense that the automorphisms of the E_n operad is the unipotent group arising from cohomology of simple graphs.

1.4. **Summary.**

- Many problems in mathematics can be restated in terms of graph complexes.
- Graph complexes offer "clean" presentations of these problems.
- But one should remember that they are not always the best for solving the problem.
- One could do research in the following ways:
  (1) Reduce other problems in mathematics to graph complex computation.
  (2) Computing graph cohomology.
  (3) Zoology: relate different graph complexes and classify them in terms of difficulty.

2. Combinatorial definition of the simplest graph complex

Let $\mathrm{gra}_{r,k}^{\geq 3,\mathrm{conn}}$ denote the set of connected, directed graphs Γ with vertex set $\{1,\dots,r\}$ and k edges (ordered and directed), i.e., pairs $(i_1,j_1),\dots,(i_k,j_k)$, such that all vertices have valency ≥ 3. Denote by S_n the symmetric group in n letters. Then

$$S_r \times (S_k \ltimes S_2^k)$$

acts on $\mathrm{gra}_{r,k}^{\geq 3,\mathrm{conn}}$ by permuting vertex labels, permuting edge labels and flipping edge directions respectively. Let us fix $n \in \mathbb{Z}$ and define the degree of $\Gamma \in \mathrm{gra}_{r,k}^{\geq 3,\mathrm{conn}}$ by

$$\deg(\Gamma) := k(n-1) - (r-1)n,$$

so that edges have degree $n-1$ and vertices have degree $-n$.

Definition 2.1. The underlying graded space of the graph complex is

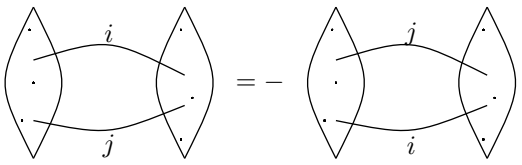
$$\mathrm{G}_n := \bigoplus_{r,k} \left(\mathbb{Q}\langle \mathrm{gra}_{r,k}^{\geq 3,\mathrm{conn}} \rangle \otimes \mathrm{twist}_{n,k,r} \right)_{S_r \times S_k \ltimes S_2^k},$$

where the subscript $S_r \times S_k \ltimes S_2^k$ means taking coinvariants, and

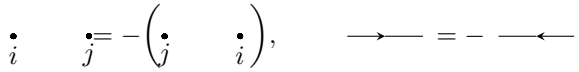
$$\mathrm{twist}_{n,k,r} := \begin{cases} \mathrm{sgn}_k & \text{if } n \text{ is even ("edges are odd")}, \\ \mathrm{sgn}_r \otimes \mathrm{sgn}_2^{\otimes k} & \text{if } n \text{ is odd ("vertices are odd")}. \end{cases}$$

Example 2.2. Here are the examples of the above sign conventions.

n even:

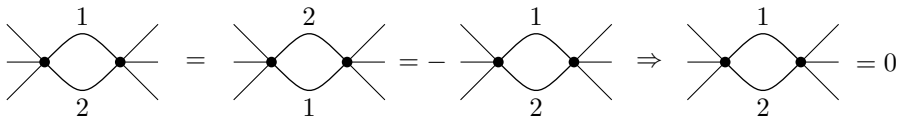


n odd:

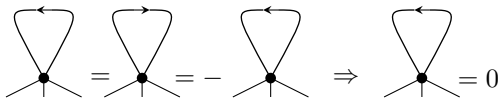


Remark 2.3. The signs already give us some information.

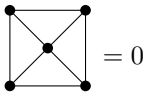
n even:



n odd:



Exercise 2.4. Show that

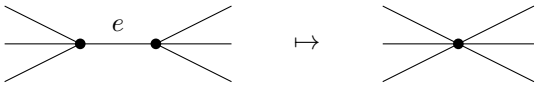


for any n.

Definition 2.5. The differential on G_n is defined as

$$d\Gamma = \sum_e \pm \Gamma/e,$$

where the summation is over the non-loop edges of Γ. Recall that by Γ/e we mean



where the sign and numbering on Γ/e is defined as follows.

n even: If e is the j-th in the ordering,
- $\pm = (-1)^{j+1}$,
- keep ordering on the other edges.

Note that the numbering on vertices doesn't matter since we take S_r coinvariants.

n odd: Suppose e is an edge from vertex i to j, with $i \neq j$. Then
- $\pm = \begin{cases} (-1)^{i+j+1} & \text{if } i < j \\ (-1)^{i+j} & \text{if } i > j \end{cases}$
- Numbering of vertices is unchanged, except that the newly formed vertex becomes the first.

Exercise 2.6. Show that d is well-defined, and satisfies $d^2 = 0$.

Remark 2.7. The complex just described admits a structure of differential graded Lie algebra.

3. Knot theory, or rational homotopy type of little disks operad

3.1. **Operads.** Operads offer a high level perspective of algebraic structure in the following sense. It is an attempt to describe the parallels in different algebraic structures. Before, every deformation/homotopy theory had to be done separately for different algebraic structures. Moving up one level gives a universal answer. More precisely, an operad encodes the "space of operations" on some type of algebraic objects.

Example 3.1. Let A be an associative algebra over $\mathbb{Q}$ (or some field k), and suppose $a_1, \dots, a_n$ are elements of A. Then the most form of operations on these to obtain an element of A are expressed as

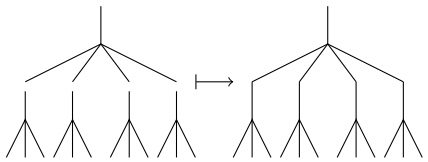
$$(a_1, \dots, a_n) \mapsto \sum_{\sigma \in S_n} \lambda_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)}$$

for some numbers $\lambda_\sigma \in \mathbb{Q}$. So the operad $\mathsf{As} = (\mathsf{As}(n))_n$ encoding this structure is given by the regular representations $\mathsf{As}(n) = \mathbb{Q}[S_n]$.

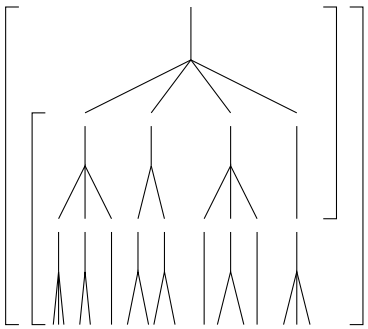
Definition 3.2. An *operad* P is a collection $\mathsf{P}(r)$ ($r = 0, 1, \dots$) of right S_r-modules, together with
- composition morphisms
$$\mu_{m,r_1,\dots,r_m} : \mathsf{P}(m) \otimes \mathsf{P}(r_1) \otimes \dots \otimes \mathsf{P}(r_m) \longrightarrow \mathsf{P}(r_1 + \dots + r_m)$$
- a unit element $1 \in \mathsf{P}(1)$,

subject to the axioms
- "associativity" of composition
- the unit axioms $\mu_1(1; x) = x$ and $\mu_m(x; 1, 1, \dots, 1) = x$ (here we suppressed the other indices).
- Equivariance of composition under the S_m-action.

Pictorially, we can view the composition as



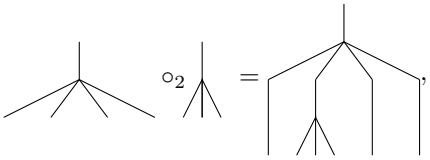
and in the same way the associativity axiom should become clear as equality of the two different ways of composing the following (starting at the bottom or starting at the top)



Let us introduce some convenient variations/notations. First, the composition is characterized by maps

$$\circ_j \colon \mathsf{P}(m) \otimes \mathsf{P}(l) \longrightarrow \mathsf{P}(m+l-1)$$

given by $x \circ_j y = \mu_m(x; 1, \dots, 1, y, 1, \dots, 1)$, with y after the jth 1. Pictorially we have for instance



note that the $\circ_j$ completely determine the compositions.

It is often useful to label inputs by sets. So, instead of the collection $\mathsf{P}(r)$, we consider a functor

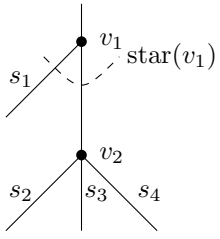
$$\mathsf{P} \colon (\mathbf{FinSets}, \mathrm{bij})^{\mathrm{op}} \to \mathbf{dgVect}$$

from the opposite category of finite sets and bijections to the one of differential graded vector spaces. Note the symmetric group actions and equivariance are now implied by looking at the bijections. We recover the previous by $\mathsf{P}(r) = \mathsf{P}(\{1, \dots, r\})$.

With the above extension, we can now formulate the composition in terms of trees. Consider a tree T whose leaves are labeled by elements of some set S. We define the tree-wise tensor product as

$$\bigotimes_T \mathsf{P} := \bigotimes_v \mathsf{P}(\mathrm{star}(v)),$$

where v ranges over the vertices of T.



The composition of P can be formulated as the "concatenation" map $\mu_T \colon \bigotimes_T \mathsf{P} \to \mathsf{P}(S)$.

Definition 3.3. We will also consider the dual concept of a *cooperad* C, whose structure is encoded by maps

$$\Delta_T \colon \mathsf{C}(S) \longrightarrow \bigotimes_T \mathsf{C}$$

when T and S are as above.

Remark 3.4. We can also make sense of operads taking value in any symmetric monoidal category, such as:

(1) differential graded vector spaces,
(2) topological spaces,
(3) operads in differential graded coalgebras, cooperads in differential graded algebras or (these are called dg *Hopf operads* and dg *Hopf cooperads* respectively),
(4) cooperads in differential graded commutative algebras, ... (commutative dg Hopf cooperads, ...).

Let us list the most fundamental examples of algebraic structures and corresponding operads.

Associative algebras: $\mathsf{As}(r) = \mathbb{Q}[S_r]$, so that $\mathsf{As}(S) = \mathbb{Q}[\mathrm{Sym}(S)]$ for any finite set S. The non-unital associative algebras correspond to the convention $\mathsf{As}(0) = 0$.

Commutative algebras: $\mathsf{Com}(r) = \mathbb{Q}$ for all r.

Lie algebras: $\mathsf{Lie}(0) = 0$ and $\mathsf{Lie}(r) = \mathbb{Q}^{(r-1)!}$, with basis $[x_{\sigma(1)}, [x_{\sigma(2)}, \dots, [x_{\sigma(n-1)}, x_n] \dots]]$. In terms of generators and relations, $\mathrm{Lie}(r)$ is given by $[\cdot,\cdot] \in \mathsf{Lie}(2)$ and the Jacobi relations.

Poisson n-algebras: Pois_n is generated by $\cdot \wedge \cdot \in \mathsf{Pois}_n(2)$ of degree 0, $[\cdot,\cdot] \in \mathsf{Pois}_n(2)$ of degree $1-n$ with the relations of a commutative product for $\cdot \wedge \cdot$, the relations of a (grade) Lie bracket for $[\cdot,\cdot]$ and the compatibility relation

$$[x_1, x_2 \wedge x_3] = [x_1, x_2] \wedge x_3 + [x_1, x_3] \wedge x_2$$

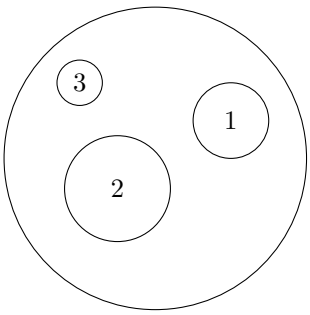
Exercise 3.5. Check that $\dim \mathsf{Pois}_n(r) = r!$.

Shifting: Given $m \in \mathbb{Z}$, every operad P induces a new operad by

$$\mathsf{P}\{m\}(r) = \begin{cases} \mathsf{P}(r)[(r-1)m] & \text{if } m \text{ is even,} \\ \mathsf{P}(r)[(r-1)m] \otimes \mathrm{sgn} & \text{if } m \text{ is odd.} \end{cases}$$

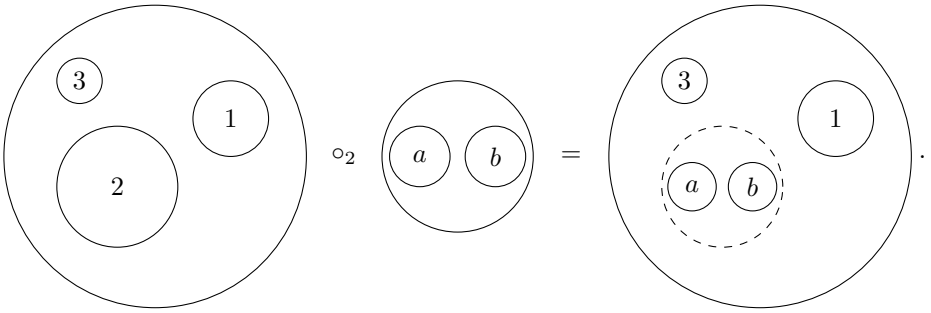
This is chosen such that if V is a P-algebra, then $V[m]$ is a $\mathsf{P}\{m\}$ algebra.

Little Disks Operad: $\mathsf{LD}_n(r)$ is the space of rectilinear embeddings of r "small" disks in the n dimensional unit disk. Here 'rectilinear' means that you can scale and translate the disks, but you cannot deform, rotate or do anything else to them. For example,

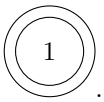


is an element of $\mathsf{LD}_2(3)$. The action of S_r on $\mathsf{LD}_n(r)$ is given by permuting the little disks, and the composition $\circ_j$ is given by gluing into the j-th disk and forgetting about

the boundary of the jth disk. For example



The unit is given by the an n-disk with single n-disk embedded in it, for example



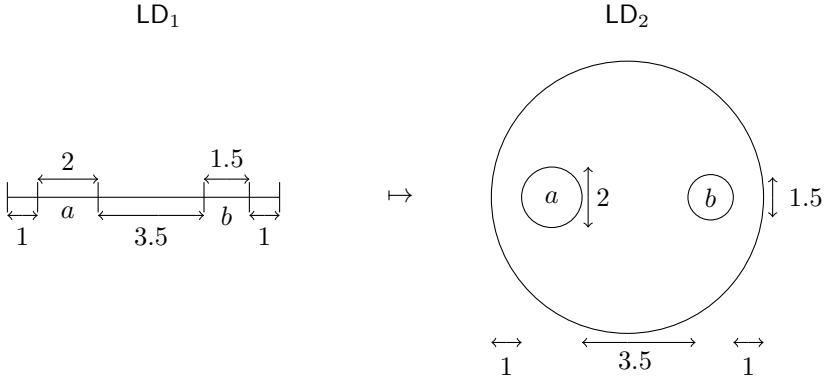
Furthermore we have $\mathsf{LD}_n(0) = \star$ and composition at j with $\star$ amounts to forgetting the j-th disk.

Definition 3.6. If there is a chain of weak equivalences

$$\mathsf{LD}_n \xrightarrow{\sim} \dots \xleftarrow{\sim} \dots \xrightarrow{\sim} \dots \xleftarrow{\sim} \dots \xrightarrow{\sim} \mathsf{P}$$

from the LD_n operad to P, then P is called an E_n *operad*.

Remark 3.7. We have the "equator embedding" maps $\mathsf{LD}_n \to \mathsf{LD}_{n+1}$, which are the higher dimensional analogues of the map $\mathsf{LD}_1 \to \mathsf{LD}_2$ given by



Remark 3.8. It has been historically well studied when it arose in topology in the following ways:

- LD_n acts on $\Omega^n X$ (n-fold based loop space) for any X by "filling in little disks by loops and mapping the outside to the the base point".
- (May [May72]): If LD_n acts on a connected space X, then $\exists Y$ such that $X \simeq \Omega^n Y$ ($\simeq$ denotes weak equivalence). (This is not why we still study them today).
- (Getzler–Jones [GJ94], Kontsevich [Kon99], Salvatore [Sal01]): The little disks operads can be also described by the sequence $(\mathrm{FM}_n(r))_r$ of the Fulton–MacPherson [FM94] (or Axelrod–Singer [AS94]) compactifications of configuration spaces of points,

$$\mathrm{FM}_n(r) = \overline{\left(\mathrm{Conf}_r(\mathbb{R}^n) \Big/ (\mathbb{R}_{>0} \ltimes \mathbb{R}^n)\right)}.$$

Roughly speaking, this compactification is obtained by replacing k-points by a copy of $\mathrm{FM}_n(k)$ when they 'collide'.

Theorem 3.9 (F. Cohen [Coh76])**.** *The homology of the little disks operad is given by*

$$\mathrm{H}(\mathsf{LD}_n;\mathbb{Q}) = \begin{cases} \mathsf{As}^u & \text{if } n=1 \\ \mathsf{Pois}_n^u & \text{if } n \geq 2 \end{cases}$$

where the u that means we have a 0-ary operation (unit).

3.2. **Goodwillie–Weiss calculus.** Consider the space of embeddings $\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ of $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ which agree with the standard embedding outside of a compact ball. First we introduce some technicality. We can consider

$$\overline{\mathrm{Emb}_\partial}(\mathbb{R}^m,\mathbb{R}^n) \longrightarrow \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n) \longrightarrow \mathrm{Imm}_\partial(\mathbb{R}^m,\mathbb{R}^n)$$

where $\overline{\mathrm{Emb}_\partial}(\mathbb{R}^m,\mathbb{R}^n)$ is the homotopy fiber. Since $\mathrm{Imm}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ is well understood (homotopy equivalent to a loop space of some Grassmannian), for a topologist the space $\overline{\mathrm{Emb}_\partial}(\mathbb{R}^m,\mathbb{R}^n)$ is equally good. We shall use the following theorem as a black box. From now on, we will reserve the symbol $\simeq$ for weak equivalences. By the work of Goodwillie–Weiss [Wei99, GW99], Dwyer–Hess [DH12], and Boavida de Brito–Weiss [BdBW15] among others, the homotopy type of this homotopy fiber can be identified as follows.

Theorem 3.10. *When $n-m \geq 3$, one has*

$$\overline{\mathrm{Emb}_\partial}(\mathbb{R}^m,\mathbb{R}^n) \simeq \Omega^{n+1}\,\mathrm{Map}_{\mathbf{TopOp}}(\mathsf{LD}_m,\mathsf{LD}_n)$$

where **TopOp** *denotes the category of topological operads and Ω denotes the (based) loop space.*

Thus, we can instead study the property of $\mathrm{Map}_{\mathbf{TopOp}}(\mathsf{LD}_m,\mathsf{LD}_n)$. The rational homotopy theory tells that there is a correspondence between the topological spaces and (certain type of) dg commutative algebras. It follows that topological operads correspond to certain dg commutative Hopf cooperads.

Lemma 3.11. *Let $\mathsf{E}_n^{\mathrm{Hc}}$ be a rational differential graded Hopf cooperad model for LD_n. When $n-m \geq 2$, one has*

$$\mathrm{Map}_{\mathbf{TopOp}}(\mathsf{LD}_m,\mathsf{LD}_n)_{\mathbb{Q}} \simeq \mathrm{Map}_{\mathbf{TopOp}}(\mathsf{LD}_m,\mathsf{LD}_n^{\mathbb{Q}}) \simeq \mathrm{Map}_{\mathbf{dgHCoop}}(\mathsf{E}_n^{\mathrm{Hc}},\mathsf{E}_m^{\mathrm{Hc}}).$$

The above correspondence should be understood based on an adjunction

$$\mathbf{TopOp} \rightleftarrows \mathbf{dgHCoop},$$

where going right and then left gives the rationalisation. In the end, our goal then has become to study $\mathrm{Map}_{\mathbf{dgHCoop}}(\mathsf{E}_n^{\mathrm{Hc}},\mathsf{E}_m^{\mathrm{Hc}})$ (see Theorem 4.6). We should first describe the mapping space in the category **dgHCoop**, secondly we should describe the specific Hopf cooperads we are working with, and lastly we should do an actual computation.

3.3. **Deformation theory of operads.** We first try to find a recipe for a computable model of $\mathrm{Map}_{\mathbf{dgHCoop}}$. The philosophy is as follows. The mapping space $\mathrm{Map}(A,B)$ in a model category should be a simplicial set

$$\mathrm{Map}(A,B) = \mathrm{Hom}(\hat{A},\tilde{B}^\Delta),$$

where

- $\hat{A}$ is a cofibrant replacement of A,
- $\tilde{B}$ is a fibrant replacement of B, and
- and $\tilde{B}^\Delta$ is a simplicial frame of $\tilde{B}$.

The space is then the linearization of the induced simplicial set. In practice, we do the following.

- Find a dg Lie (or L_∞) algebra $\mathfrak{g}$ that "governs" the maps $\hat{A} \to \tilde{B}$, in the sense that the set of maps can be identified to that of Maurer–Cartan elements

$$\mathrm{MC}(\mathfrak{g}) = \left\{ x \in \mathfrak{g}^1 | dx + \frac{1}{2}[x,x] = 0 \right\}.$$

- Then represent $\mathrm{Map}(A, B)$ by the simplicial set
$$\mathrm{MC}_\bullet(\mathfrak{g}) := \mathrm{MC}(\Omega_{\mathrm{pol}}(\Delta^\bullet) \hat{\otimes} \mathfrak{g}),$$
where $\Omega_{\mathrm{pol}}(\Delta^n)$ is the set of polynomial differential forms on the standard n-simplex. Note that when $\mathfrak{g}$ is a usual Lie algebra, $\mathrm{MC}_\bullet(\mathfrak{g})$ is the space of flat connection on $\Delta^\bullet$ with values in $\mathfrak{g}$.

Theorem 3.12 (Berglund [Ber11])**.** *Suppose $\mathfrak{g}$ is a pronilpotent L_∞-algebra and m be an element of $\mathrm{MC}_\bullet(\mathfrak{g})$. Then one has*
$$\pi_k(\mathrm{MC}_\bullet(\mathfrak{g}), m) = \mathrm{H}^{1-k}(\mathfrak{g}^m),$$
where $\mathfrak{g}^m$ denotes the $\mathfrak{g}$ with the modified L_∞-structure given by
$$\mu_j^m = \sum_{k\geq 0} \frac{1}{k!} \mu_{j+k}(-, -, \dots, -, m, \dots, m).$$

Note that for $\mathfrak{g}$ is a dg Lie algebra, the structure of $\mathfrak{g}^m$ is simply given by the new differential $d + [m, -]$ and the same Lie bracket.

3.3.1. *dg operads.* Let us first consider ordinary operads as a toy model.

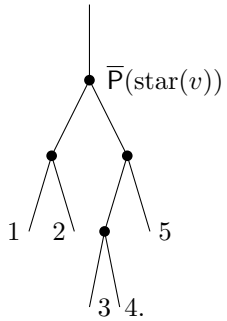
Suppose P and Q are operads in the category of differential graded vector spaces. Since Q is fibrant already, there is no need to pick a resolution (otherwise we would need to find a quasi-cofree resolution). There are essentially two ways to construct a quasi-free resolution of P:

- When P is augmented by a map $\mathsf{P} \to \star$, we use the bar-cobar construction $\Omega(B(\mathsf{P}))$.
- When P is Koszul, we may use $\Omega(\mathsf{P}^\vee)$, where $\mathsf{P}^\vee$ is the Koszul dual ($\mathsf{P}^\vee$ is weakly equivalent to $B(\mathsf{P})$, but it is simpler).

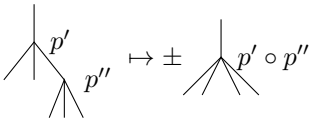
Hence the mapping space will be captured by

$$\mathfrak{g} = \mathrm{Hom}(\hat{\mathsf{P}}, \mathsf{Q}) \quad (\hat{\mathsf{P}} = \Omega(\mathsf{C}) \text{ for the cooperad } \mathsf{C} = \mathsf{P}^\vee \text{ or } B(\mathsf{P})). \tag{1}$$

More concretely, the bar construction $B(\mathsf{P})$ is a cooperad given as
$$B(\mathsf{P}) = (\mathrm{Cofree}(\bar{\mathsf{P}}[1]), \partial),$$
where $\bar{\mathsf{P}}$ is the kernel of the augmentation map $\mathsf{P} \to \star$. Here, Cofree denotes taking the cofree cooperad cogenerated by the argument. This means that we consider trees with vertices v decorated by $\overline{\mathsf{P}}(\mathrm{star}(v))$, like



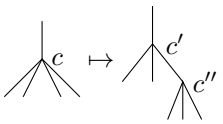
The differential is defined as $\partial = d_{\mathsf{P}} + \partial'$, where ∂' is given by summing over the edges of the tree and composing the decorations like



The cocomposition is given by cutting the trees.

The dual construction is the cobar construction Ω. Given a differential graded cooperad C (like $B(\mathsf{P})$), the cobar of C is given as
$$\Omega(\mathsf{C}) = (\mathrm{Free}(\overline{\mathsf{C}}[-1]), \partial),$$

where $\overline{\mathsf{C}} = \mathrm{Coker}(\star \to \mathsf{C})$. Again Free denotes taling the free operad generated by the argument, which means considering trees with vertices v decorated by $\overline{\mathsf{C}}(\mathrm{star}(v))$ (the same picture as for Cofree works). The differential is given as $\partial = d_{\mathsf{C}} + \partial'$, where ∂' is given by the sum over all vertices of



where c cocomposes to $\sum c' \otimes c''$. There is a canonical quasi-isomorphism $\Omega(B(\mathsf{P})) \xrightarrow{\sim} \mathsf{P}$, which is given on generators $B(\mathsf{P})$ by mapping

and killing all non-trivial trees. This is our cofibrant replacement of P, and it is always of the form $\Omega(\mathsf{C})$ for a coaugmented cooperad C ($\mathsf{C} = \mathsf{P}^\vee$ or $\mathsf{C} = B(\mathsf{P})$).

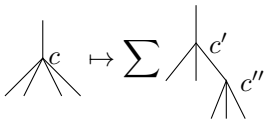
We are now trying to understand $\mathrm{MC}_\bullet(\mathfrak{g}) := \mathrm{MC}(\Omega(\Delta^\bullet)\hat{\otimes}\mathfrak{g})$, where $\mathfrak{g}$ is given by (1). Denote

$$\mathrm{Conv}\left(\mathsf{C},\mathsf{Q}\right) := \prod_r \mathrm{Hom}_{S_r}\left(\overline{\mathsf{C}}[-1](r), \mathsf{Q}(r)\right),$$

and note that for C as above this is just the definition of $\mathfrak{g}$ spelled out. In fact, $\mathrm{Conv}(\mathsf{C},\mathsf{Q})$ is a dg Lie algebra with the bracket

$$[f,g]\left(\quad c\quad\right) = \sum \mu_{\mathsf{Q}}\left(\quad f(c')\quad g(c'')\quad\right) - (-1)^{|f||g|}\left(f \leftrightarrow g\right),$$

where the cocomposition of C is expressed as



Exercise 3.13. Check that the above formula defines a Lie bracket.

The differential is given by $d = d_{\mathsf{C}} + d_{\mathsf{Q}}$. Compatibility with the differntial of $\hat{\mathsf{P}}$ and Q amounts exactly to the Maurer–Cartan equation. Suppose $F\colon \hat{\mathsf{P}} \to \mathsf{Q}$ is induced by $f\colon \overline{C} \to \mathsf{Q}$. Compatibility says $d_{\mathsf{Q}}F = Fd_{\hat{\mathsf{P}}}$, suppose $c \in C$ then we have that

$$d_{\mathsf{Q}}F(c) = d_{\mathsf{Q}}f(c)$$

and

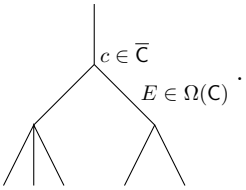
$$Fd_{\hat{\mathsf{P}}}(c) = F\left(d_C(c) + \sum \pm \quad c'\quad c''\quad\right) = f(d_C c) + \frac{1}{2}[f,f](c).$$

So we find that

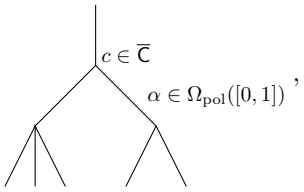
$$d_{\mathsf{Q}}F = Fd_{\hat{\mathsf{P}}} \Longrightarrow df + \frac{1}{2}[f,f] = 0.$$

3.3.2. *Hopf cooperads.* Now suppose B and C are differential graded Hopf cooperads. We need to take a cofibrant (free/quasi-free) replacement of B and a fibrant (cofree) replacement of C in the category of dg Hopf cooperads. For B we take the Chevalley complex $C(\mathfrak{g})$ (here $\mathfrak{g}$ is a cooperad in dg Lie coalgebras). Specifically, a model for $\mathsf{E}_n^{\mathrm{Hc}}$ will be given by $C(\mathsf{P}_n)$, where $\mathsf{P}_n^* = (\mathfrak{t}_n(r))_r$ is the family of Drinfeld–Kohno Lie algebras as follows: $\mathfrak{t}_n(r)$ is generated by the elements t_{ij} for $1 \le i \ne j \le r$ such that $t_{ij} = (-1)^n t_{ji}$, $[t_{ij}, t_{kl}] = 0$ when $i \ne k$ and $j \ne l$ and $[t_{ij}, t_{kl} + t_{kj}] = 0$.

For C we use the "W"-construction of Berger–Moerdijk. If $B(\Omega(\mathsf{C}))$ is gven by tree whose vertices are labeled by $c \in \overline{\mathsf{C}}$ and edges by $E \in \Omega(\mathsf{C})$, e.g.,



Then $W(\mathsf{C})$ is given by trees with vertices labeled by $c \in \overline{\mathsf{C}}$ and edges labeled by $\alpha \in \Omega_{\mathrm{pol}}([0,1]) = k[t, dt]$, e.g.,



and when we "shrink" an edge we see the cocomposition. The point is that $B(\Omega(\mathsf{C}))$ does not have a good multiplication, while $W(\mathsf{C})$ does (we are sweeping some small technical points under the rug here). So it is a cooperad in dg algebras, where the product is defined by "pointwise" mulitplication of labels on trees of the same shape. We check that $W(\mathsf{C})$ is of the form $B(\mathring{W}\mathsf{C})$ for some operad $\mathring{W}\mathsf{C}$ (given by indeomposable elements of $W(\mathsf{C})$), consequently $W(\mathsf{C})$ is fibrant.

In the end we have to analyze

$$\mathrm{Hom}_{\mathbf{dgHCoop}}(C(\mathsf{P}_n)\{1\}, W(\mathsf{C})) \subset \mathrm{Hom}_{\mathbf{Op}}(C(\mathsf{P}_n)\{1\}, W(\mathsf{C})).$$

Restricting to the generator P_n of $C(\mathsf{P}_n)$ and corestricting to the cogenerator $\mathring{W}\mathsf{C}$ of $B(\mathring{W}\mathsf{C}) = W(\mathsf{C})$, the right hand side has the same information as

$$\mathrm{HConv}\left(\mathsf{P}_n, \mathring{W}\mathsf{C}\right) := \prod_r \mathrm{Hom}_{S_r}\left(\mathsf{P}_n(r), \mathring{W}\mathsf{C}(r)\right)[1].$$

Proposition 3.14. *There is dg Lie algebra structure on* $\mathrm{HConv}\left(\mathsf{P}_n, \mathring{W}\mathsf{C}\right)$ *such that the Maurer–Cartan elements are in one to one correspondence with dg Hopf cooperad maps from* $C(\mathsf{P}_n)$ *to* $W(\mathsf{C})$.

So (given this proposition holds) we have found our dg Lie algebra and we now need the rational models for E_n.

Theorem 3.15 ([FW15])**.** *The little disks operads* LD_n *are rationally formal, i.e., we may take the cohomology* $\mathrm{H}^*(\mathsf{LD}_n, \mathbb{Q}) = \mathsf{e}_n^*$ *as a rational dg Hopf cooperad model for* LD_n.

Let us note that several versions of this theorem are already known.

$n = 1$: trivial
$n = 2$: Tamarkin [Tam03] (using a rational Drinfeld associator)
$n \geq 2$ *over* $\mathbb{R}$: Kontsevich [Kon99]
$n \geq 2$ *without* 0*-ary operations*: Deduced from the previous by Salvatore
$n \geq 3$: (with 0-ary operations which are replaced by a Λ-structure) Fresse–Willwacher.

If 4 divides n, there is an involution J on C modeling the "reflection" on e_n^*.

Theorem 3.16 ([FW15])**.** *For* $n \geq 3$ *we have intrinsic formality over* $\mathbb{Q}$*, that is, if* C *is a dg Hopf cooperad such that its cohomology* $\mathrm{H}^*(\mathsf{C})$ *is isomorphic to* e_n^**, then we actually have* $\mathsf{C} \simeq \mathsf{e}_n^*$ *in the homotopy category of dg Hopf cooperads. This morphism is:*

- *homotopically unique if* 4 *does not divide* $n - 3$,
- *unique in presence of an involution if* 4 *divides* $n - 3$,

The comparison of $C(\mathsf{P}_n)$ and e_n^* goes as follows: (assuming $n \geq 2$) we look at the dual statement, and consider the map

$$\mathsf{e}_n \to C(\mathfrak{t}_n), \quad \cdot \wedge \cdot \mapsto 1 \in C(\mathfrak{t}_n(2)) = \mathrm{Sym}(\mathfrak{t}_n(2)[1]), \quad [\cdot,\cdot] \mapsto t_{12}.$$

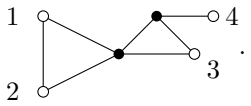
By induction on r, we obtain $\mathrm{H}(\mathfrak{t}_n(r)) \simeq \mathsf{e}_n(r)$ so that the above is a quasi-isomorphism.

4. Connection to graph complexes

So let us show how to obtain graph complexes from the situation above. For simplicity we consider the operad version first. The main players are the operad Graphs_n [1] and its dual cooperad ${}^*\mathsf{Graphs}_n$ (so that $({}^*\mathsf{Graphs}_n)^* = \mathsf{Graphs}_n$ holds) introduced by Kontsevich. More precisely, ${}^*\mathsf{Graphs}_n(r)$ is given by k-linear combinations of isomorphism classes of graphs with r numbered "external" vertices and ≥ 0 (unnumbered) "internal" vertices, such that

- internal vertices have valency ≥ 3, and
- every connected component has ≥ 1 external vertex.

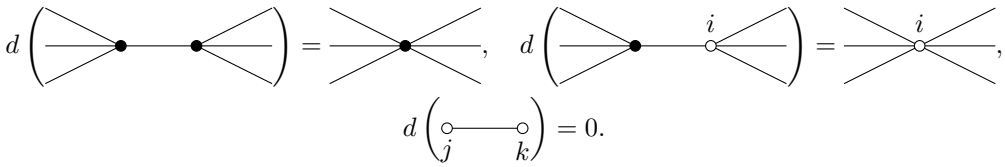
For example, we have the following graph:



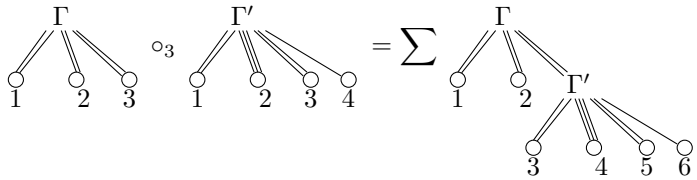
We set degree of a graph by

$$\deg(\Gamma) = (\text{number of internal vertices})(-n) + (\text{number of edges})(n-1).$$

The differential is given by



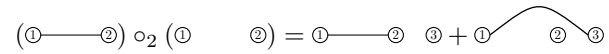
Let us define an operad structure on Graphs_n. The composition $\Gamma \circ_r \Gamma'$ on Graphs_n is given by removing the r-th vertex of Γ and summing over all ways of reconnecting the pending edges, i.e.,



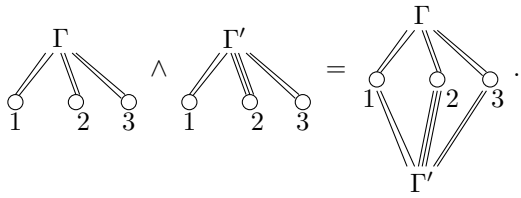
Example 4.1. Consider the map $\mathsf{Pois}_n \to \mathsf{Graphs}_n$ characterized by

$$\cdot \wedge \cdot \mapsto \textcircled{1} \qquad \textcircled{2}, \quad [\cdot,\cdot] \mapsto \textcircled{1}\!\!-\!\!\textcircled{2}.$$

We have compatibility $[x_1, x_2 \wedge x_3] = [x_1, x_2] \wedge x_3 + [x_1, x_3] \wedge x_2$ since



There is a Hopf structure on ${}^*\mathsf{Graphs}_n$ induced by identifying external vertices, i.e



This makes ${}^*\mathsf{Graphs}_n$ into a Hopf cooperad.

[1] Not to be confused with the more elementary graph operad Gra behind the complexes G_n.

Theorem 4.2 (Kontsevich [Kon99], Lambrechts–Volić [LV14])**.** *The maps*

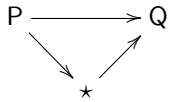
$$\mathsf{Pois}_n \to \mathsf{Graphs}_n, \quad {}^*\mathsf{Graphs}_n \to \mathsf{Pois}_n^*$$

are quasi-isomorphisms of dg Hopf (co)operads.

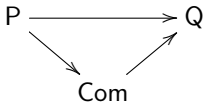
We actually need to understand the Hopf operad maps, but for simplicity let's just look at the operad maps. Note that e_m (either As or Pois_n) is Koszul and its Koszul dual is given by $\mathsf{e}_m^\vee \simeq \mathsf{e}_m^*\{m\}$ (see for example [LV12, Chapter 7]). Then $\Omega(\mathsf{e}_m^\vee)$ is a cofibrant operad model for $\mathsf{E}_m^{\mathrm{Hc}}$ (or for $\mathrm{H}(\mathsf{LD}_m)$), and Graphs_n is a (automatically fibrant) operad model for $\mathsf{E}_n^{\mathrm{Hc}}$ (or for $\mathrm{H}(\mathsf{LD}_n)$). We thus look at $\Omega(\mathsf{e}_m^\vee) \to \mathsf{Graphs}_n$, and study

$$\mathrm{Conv}(\mathsf{e}_m^\vee, \mathsf{Graphs}_n) = \prod_r \mathrm{Hom}_{S_r}\left(\mathsf{e}_m^\vee(r), \mathsf{Graphs}_n(r)\right) \simeq$$
$$\prod_r \left(\mathsf{e}_m^\vee\right)^*(r) \otimes_{S_r} \mathsf{Graphs}_n(r) = \prod_r \mathsf{e}_m\{m\}(r) \otimes_{S_r} \mathsf{Graphs}_n(r).$$

Remark 4.3. In the Hopf operadic setting there are no simple factorizations like



since each $\mathsf{P}(r)$ and $\mathsf{Q}(r)$ should be a unital algebra. Instead we consider factorizations of the form



using the commutative algebra operad Com (recall that $\mathsf{Com}(r) = \mathbb{Q}$ for all r).

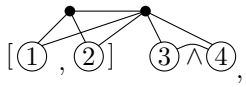
Now we can twist by an Maurer–Cartan element α corresponding to

$$\star\colon \Omega(\mathsf{e}_m^\vee) \to \mathsf{Com} \to \mathsf{Graphs}_n,$$

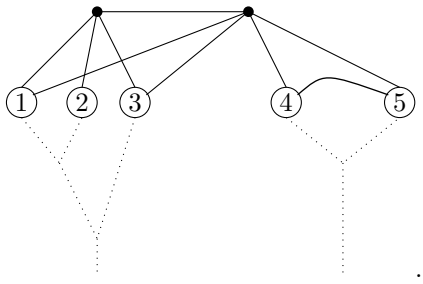
to obtain the deformation complex

$$\mathrm{Def}(\Omega(\mathsf{e}_m^\vee) \xrightarrow{\star} \mathsf{Graphs}_n) := \mathrm{Conv}(\mathsf{e}_m^\vee, \mathsf{Graphs}_n)^\alpha.$$

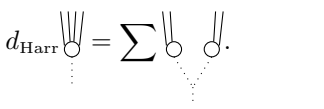
Elements of $\mathrm{Conv}(\mathsf{e}_m^\vee, \mathsf{Graphs}_n)$ are series of "graphs" with Gerstenhaber expression on the external vertices, like



or Lie forests describing composition of $\cdot \wedge \cdot$ and $[\cdot,\cdot]$:



With the differential $d = d_{\mathsf{Graphs}_n} + d_{\mathrm{Harr}}$ where the Harrison differential d_{Harr} is given by



Theorem 4.4 ("Hochschild–Kostant–Rosenberg"). *Let $m \geq 2$, and consider the subscomplex*

$$\mathrm{fHGC}_{m,n} \subset \mathrm{Def}\left(\Omega(\mathsf{e}_m^\vee) \xrightarrow{\star} \mathsf{Graphs}_n\right)$$

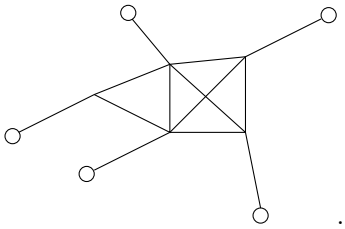
spanned by graphs as above, such that

(1) *external edges all have valency* 1*, and*
(2) *Lie trees are all trivial.*

Then the inclusion is a quasi-isomorphism, and $\mathrm{fHGC}_{m,n}$ *is a sub dg Lie algebra.*

It follows that we just need to study $\mathrm{MC}_\bullet(\mathrm{fHGC}_{m,n})$.

Note that the subcomplex we are dealing with is just given by simple graphs like



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So we have reduced the problem to a graph complex problem.

4.1. **Obtaining information on graph cohomology.** Recall that

$$\mathrm{fHGC}_{m,n} \xrightarrow{\sim} \mathrm{Def}(\Omega(\mathsf{e}_m^\vee) \xrightarrow{\star} \mathsf{Graphs}_n) = \mathrm{Conv}(\mathsf{e}_m^\vee, \mathsf{Graphs}_n)^\alpha,$$

where the map $\star$ is the map factoring through Com and α is the Maurer–Cartan element corresponding to $\star$. We have identification of S-modules $\mathsf{e}_m^\vee = \mathsf{Pois}_m$ (for $m \geq 2$) and $\mathsf{e}_1^\vee = \mathsf{As}$, in which the bracket becomes 0 under $\star$.

For $m \geq 2$, $\mathrm{fHGC}_{m,n}$ is a sub dg Lie algebra with bracket given by

$$\left[\Gamma, \tilde{\Gamma}\right] = \sum \Gamma \; \tilde{\Gamma} \; \mp (\Gamma \leftrightarrow \tilde{\Gamma}),$$

where the sum runs over all ways of connecting the "hairs".

Remark 4.5. For $m = 1$, the Hochschild–Kostant–Rosenberg type theorem gives a quasi-isomorphism between T_{poly} (polyvector fields) and D_{poly} (polydifferential operators) which is a priori not compatible with the Lie bracket. By Kontsevich's theorem it does extend to an L_∞ morphism. However, $T_{\mathrm{poly}}^{\geq 1}$ and $D_{\mathrm{poly}}^{\geq 1}$ are not L_∞ quasi-isomorphic (Dolgushev–Tamarkin–Tsygan). In fact, $T_{\mathrm{poly}}^{\geq 1}$ admits a one-parameter family of inequivalent L_∞ structures. The Shoikhet L_∞ structure is the one compatible with $D_{\mathrm{poly}}^{\geq 1}$, and transferring this structure to $\mathrm{fHGC}_{1,n}$, the inclusion to $\mathrm{Def}(\Omega(\mathsf{As}) \to \mathsf{Graphs}_n)$ becomes an L_∞ quasi-isomorphism.

4.2. **Hopf cooperad case.** We are of course actually interested in the Hopf cooperad case.

Theorem 4.6 ([FW15]). *Let* HGC *denote the subcomplex of* fHGC *spanned by the connected graphs. There exists an L_∞ quasi-isomorphism*

$$\mathrm{HGC}_{m,n} \xrightarrow{\sim} \mathrm{HConv}\left(\mathsf{P}_n, \mathring{W}\mathsf{e}_m\right)$$

when $m \geq 2$, and an analogous one with respect to the "Shoikhet" structure for $m = 1$.

So we find that

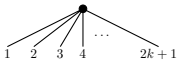
$$\mathrm{Map}\left(\mathsf{E}_n^{\mathrm{Hc}}, \mathsf{E}_m^{\mathrm{Hc}}\right) \simeq \mathrm{MC}_\bullet\left(\mathrm{HGC}_{m,n}\right),$$

where for $m = 1$ we consider the Shoikhet L_∞ structure.

4.3. **Space of the Maurer–Cartan elements.** Now we have reduced the computation to considering the graph complex $\mathrm{HGC}_{m,n}$. For the space of Maurer–Cartan elements of this complex we can say the following things.

(1) For $\mathrm{HGC}_{m,n}$ with $n - m \geq 2$ we proceed by degree counting. Since the degree of genus(=loop order) g graphs is smaller than or equal to $-g(n-3)-1$ we find that there is nothing (except for the trivial one) in degree 1 of $\mathrm{MC}\left(\mathrm{HGC}_{m,n}\right)$, where the 'base point' is supposed to be. Moreover, $\mathrm{MC}_\bullet\left(\mathrm{HGC}_{m,n}\right)$ is simply connected, since by Theorem 3.12 one has
$$\pi_k\left(\mathrm{MC}_\bullet\left(\mathrm{HGC}_{m,n}\right)\right) \simeq \mathrm{H}^{1-k}\left(\mathrm{HGC}_{m,n}\right).$$

(2) For $\mathrm{HGC}_{n,n}$ with $n \geq 2$, we have the "easy" Maurer–Cartan elements coming from
$$L = \lambda \circ\!\!-\!\!\circ,$$
where λ is a scalar parameter in $\mathbb{Q}$ and $\circ\!\!-\!\!\circ$ is given by the bracket in e_n. This works because
$$\delta \circ\!\!-\!\!\circ = 0 \ \text{ and } \ [\circ\!\!-\!\!\circ, \circ\!\!-\!\!\circ] = 0.$$
So we can define the locally constant function $F\colon \mathrm{MC}_\bullet(\mathrm{HGC}_{n,n}) \to \mathbb{Q}$ by "picking" the coefficient of the graph $\circ\!\!-\!\!\circ$. Thus we should study $F^{-1}(\lambda)$ for various values of λ.

(3) For $\mathrm{HGC}_{n-1,n}$ and $n \geq 3$, we have similarly the Maurer–Cartan elements
$$T_\lambda = \sum_{k\geq 1} \frac{\lambda^k}{(2k+1)!}$$


for $\lambda \in \mathbb{Q}$.

Exercise 4.7. Show that T_λ is a Maurer–Cartan element.

Again we construct locally constant functions $J\colon \mathrm{MC}_\bullet(\mathrm{HGC}_{n-1,n}) \to \mathbb{Q}$ by picking out the coefficient of  . So we want to study also $J^{-1}(\lambda)$.

Theorem 4.8 (Fresse–Turchin–Willwacher [FTW15])**.** *For any $\lambda \neq 0$, there are L_∞ quasi-isomorphisms*

- $\mathbb{Q}[-1] \oplus \mathrm{H}(\mathrm{GC}_n^{\geq 2})[-1] \xrightarrow{\sim} \mathrm{HGC}_{m,n}^{\lambda \frown}$ *for $n \geq 2$,*
- $\mathbb{Q}[-1] \oplus \mathrm{H}(\mathrm{GC}_n^{\geq 2})[-1] \xrightarrow{\sim} \mathrm{HGC}_{n-1,n}^{T_\lambda}$ *for $n \geq 3$,*
- $\mathbb{Q}[-1] \oplus \mathrm{H}(\mathrm{GC}_n^{\geq 2})[-1] \xrightarrow{\sim} \mathrm{HGC}_{1,2}^{T'_\lambda}$*, where the right hand side has the Shoikhet L_∞ structure.*

Here, the left hand side is considered as a trivial Lie algebra, and the complex $\mathrm{GC}_n^{\geq 2}$ denotes $\left(\mathrm{G}_n^{\geq 2}\right)^*$, where $\mathrm{G}_n^{\geq 2}$ is given by graphs with valency greater than 2.

Corollary 4.9. *For $\lambda \neq 0$, we have*
$$F^{-1}(\lambda) \simeq \mathrm{MC}_\bullet\left(\mathrm{H}(\mathrm{GC}_n^{\geq 2})\right) \simeq J^{-1}(\lambda).$$

Remark 4.10. Description of $J^{-1}(0)$ and $F^{-1}(0)$ is still an open problem.

Corollary 4.11. *If $n > 2$, when 4 does not divide $n+1$ we have*
$$\pi_0\left(\mathrm{MC}_\bullet\left(\mathrm{H}(\mathrm{GC}_n^{\geq 2})\right)\right) = 0,$$
and when 4 does divide $n+1$ we have
$$\pi_0\left(\mathrm{MC}_\bullet\left(\mathrm{H}(\mathrm{GC}_n^{\geq 2})\right)\right) = \mathbb{Q}.$$
If $n = 2$, we have
$$\pi_0\left(\mathrm{MC}_\bullet\left(\mathrm{H}(\mathrm{GC}_2^{\geq 2})\right)\right) = \mathfrak{grt}_1$$
where $\mathfrak{grt}_1$ denotes the Grothendieck–Teichmüller Lie algebra.

Theorem 4.12. *We have the following isomorphisms:*
$$\mathrm{H}^0(\mathrm{GC}_2) \simeq \mathfrak{grt}_1, \quad \mathrm{H}(\mathrm{GC}_2, \delta + [\bigcirc_\bullet, -]) \simeq \mathbb{Q}.$$

Remark 4.13.
- There is an embedding of the free Lie algebra $\mathrm{Lie}(\sigma_3, \sigma_5, \ldots)$ into $\mathfrak{grt}_1$, and conjecturally this is isomorphism.
- The comparison of the above two cohomology groups means that the "loop classes" can cancell some of the $\mathfrak{grt}_1$-classes via the commutator.
- The remaining $\mathfrak{grt}_1$-classes correspond to other Maurer–Cartan elements like

$$\sum_k \frac{1}{(2k+1)!}$$  $(2k+1 \text{ edges})$

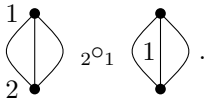
in a similar way. Thus, we can capture the negative degree classes.

5. Connections among the graph complexes of various type

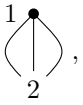
This last part is based on joint work with Merkulov [MW15]. Consider the space of cyclic words $\oplus_n V^{\otimes n}/C_n$ with letters from some vector space V. Such structure is "governed" by the prop(erad) of the ribbon graphs

$$\mathsf{RGra}(m,n) = \left\langle \right.$$  $$\left. : m \text{ vertices, } n \text{ holes} \right\rangle + \text{sign rule.}$$

The composition rule is given as follows: suppose we want to compute



The left graph becomes



which goes into the "hole 1" of the right graph, where we sum over the all possible ways to connect the free edges to vertices:

$$\sum \pm$$ 

By Chas–Sullivan [CS02], the properad of involutive Lie bialgebras map into RGra by

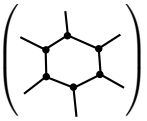
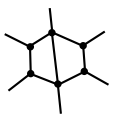
$$\mathsf{Lieb}^\diamond \to \mathsf{RGra}, \quad \mathrm{bra} \; \Big\curlyvee \mapsto$$  $$, \quad \mathrm{cobra} \; \Big\curlyvee \mapsto$$  $$.$$

Remark 5.1. We also have a map from the properad of Lie bialgebras $*\colon \mathsf{Lieb} \to \mathsf{RGra}$, given by

$$\mathrm{bra} \; \Big\curlyvee \mapsto$$  $$, \quad \mathrm{cobra} \; \Big\curlyvee \mapsto 0.$$

The corresponding deformation complex computes $\oplus \mathrm{H}^*(\mathcal{M}_{g,n}) \otimes_{S_n} \mathrm{sgn}_n$. Moreover, sgn_n can be removed by changing the degree convention in Lieb (cf. Kontsevich).

If we look at the properad map $\mathsf{Lieb}^\diamond \to \mathsf{RGra}$ given by the Chas–Sullivan correspondence, we have

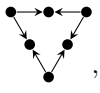
$$\mathrm{Def}(\mathsf{Lieb}^\diamond \to \mathsf{RGra}) = (\mathrm{RGC}, \delta + D), \quad D \left($$  $$\right) = \sum$$ 

Note that the right hand side will be a quotient of the cohomology of moduli spaces.

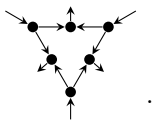
Remark 5.2. The deformation complex

$$\mathrm{Def}(\mathsf{Lieb}^\diamond \xrightarrow{\mathrm{Id}} \mathsf{Lieb}^\diamond) \simeq \mathrm{GC}_1^{\mathrm{ori}} \simeq \mathrm{GC}_0$$

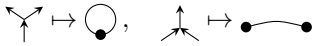
acts on $\mathrm{Def}(\mathsf{Lieb}^\diamond \to \mathsf{RGra})$. Starting from the classes in $\mathrm{GC}_1^{\mathrm{ori}}$, which are like



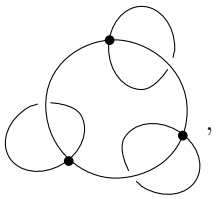
and one adds "hairs" to make it look like



The directed graph without cycles can be interpreted as composition rule in a properad. Moreover, the correspondence



gives a ribbon graph, so for example the above graph gives

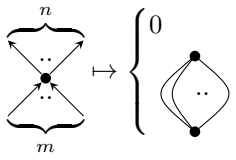


which is a cocycle in $(\mathrm{RGC}, \delta + D)$.

Conjecture 5.3. The graph cohomology $\mathrm{H}(\mathrm{GC}_0)$ embeds into $\mathrm{H}(\mathrm{RGC}, \delta + D)$. Or, even more, this is almost surjective.

Note that, if we instead consider the target $\mathrm{Def}(\mathsf{Lieb} \to \mathsf{RGra})$, this would be the case since goes to 0.

5.1. **Final remarks.** With the sign convention of Kontsevich, there is a map of properads $\mathsf{Lieb}_\infty^{\mathrm{odd}} \to \mathsf{RGra}$ given by



$$\mapsto \begin{cases} 0 & (m \neq 2), \\ \text{(} n \text{ edges, if } m = 2 \text{ and } n \text{ odd)}. \end{cases}$$

This would give new Maurer–Cartan elements.

As for the Chas–Sullivan map itself, $\mathrm{Def}(\mathsf{Lieb}^\diamond \to \mathsf{RGra})$ is given by $(\mathrm{RGC}[\![\hbar]\!], \delta + D + \tilde{D}_\hbar)$, where $\tilde{D}_\hbar$ is given by adding edges to connect holes.

Conjecture 5.4. The complex $(\mathrm{RGC}[\![\hbar]\!], \delta + D + \tilde{D}_\hbar)$ is essentially acyclic.

There is a "stable" version of the ribbon graph complex (sRGC, δ) which captures the compactification of the moduli.

Theorem 5.5. *One has the isomorphism* $\mathrm{H}(\mathrm{sRGC}) \simeq \mathrm{H}(\mathrm{RGC}) \oplus \mathrm{H}(\mathrm{GC}_0)$.

References

[AS94] S. Axelrod and I. M. Singer, *Chern-Simons perturbation theory. II*, J. Differential Geom. **39** (1994), no. 1, 173–213. MR1258919 (95b:58163)

[Ber11] A. Berglund, *Rational homotopy theory of mapping spaces via Lie theory for L_∞ algebras.* preprint (2011), available at `arXiv:1110.6145 [math.AT]`, to appear in Homology, Homotopy and Applications.

[BdBW15] P. Boavida de Brito and M. S. Weiss, *Spaces of smooth embeddings and configuration categories.* preprint (2015), available at `arXiv:1502.01640 [math.AT]`.

[CS02] M. Chas and D. Sullivan, *Closed string operators in topology leading to lie bialgebras and higher string algebra.* preprint (2002), available at `arXiv:math/0212358 [math.GT]`.

[Coh76] F. Cohen, *The homology of $\mathscr{C}_{n+1}$-spaces, $n \geq 0$*, The homology of iterated loop spaces (F. R. Cohen, T. J. Lada, and J. P. May, eds.), Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin-New York, 1976, pp. 207–351. MR0436146 (55 #9096)

[Dol11] V. Dolgushev, *Stable formality quasi-isomorphisms for Hochschild cochains I.* preprint (2011), available at `arXiv:1109.6031 [math.KT]`.

[DW15] V. Dolgushev and T. Willwacher, *Operadic twisting—with an application to Deligne's conjecture*, J. Pure Appl. Algebra **219** (2015), no. 5, 1349–1428, `DOI:10.1016/j.jpaa.2014.06.010`. MR3299688

[DH12] W. Dwyer and K. Hess, *Long knots and maps between operads*, Geom. Topol. **16** (2012), no. 2, 919–955, DOI:10.2140/gt.2012.16.919. MR2928985

[FTW15] B. Fresse, V. Turchin, and T. Willwacher, *Mapping spaces of E_n operads* (2015), in preparation.

[FW15] B. Fresse and T. Willwacher, *The intrinsic formality of E_n-operads*. preprint (2015), available at arXiv:1503.08699 [math.AT].

[FM94] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 1, 183–225, DOI:10.2307/2946631. MR1259368 (95j:14002)

[GJ94] E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*. preprint (1994), available at arXiv:hep-th/9403055 [hep-th].

[GW99] T. G. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory. II*, Geom. Topol. **3** (1999), 103–118 (electronic), DOI:10.2140/gt.1999.3.103. MR1694808 (2000c:57055b)

[Kon03] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216, DOI:10.1023/B:MATH.0000027508.00421.bf, arXiv:q-alg/9709040 [math.QA]. MR2062626 (2005i:53122)

[Kon99] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. **48** (1999), no. 1, 35–72, DOI:10.1023/A:1007555725247. Moshé Flato (1937–1998). MR1718044 (2000j:53119)

[LV14] P. Lambrechts and I. Volić, *Formality of the little N-disks operad*, Mem. Amer. Math. Soc. **230** (2014), no. 1079, viii+116. MR3220287

[LV12] J.-L. Loday and B. Vallette. (2012). *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, DOI:10.1007/978-3-642-30362-3, ISBN 978-3-642-30361-6. MR2954392

[May72] J. P. May. (1972). *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York. Lectures Notes in Mathematics, Vol. 271. MR0420610 (54 #8623b)

[MW15] S. Merkulov and T. Willwacher, *Props of ribbon graphs, involutive lie bialgebras and moduli spaces of curves*. preprint (2015), available at arXiv:1511.07808 [math.QA].

[Pen88] R. C. Penner, *Perturbative series and the moduli space of Riemann surfaces*, J. Differential Geom. **27** (1988), no. 1, 35–53. MR918455 (89h:32045)

[Vog06] K. Vogtmann, *The cohomology of automorphism groups of free groups*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1101–1117. MR2275637 (2007k:20090)

[Sal01] P. Salvatore, *Configuration spaces with summable labels*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 375–395. MR1851264 (2002f:55039)

[Tam03] D. E. Tamarkin, *Formality of chain operad of little discs*, Lett. Math. Phys. **66** (2003), no. 1-2, 65–72, DOI:10.1023/B:MATH.0000017651.12703.a1. MR2064592 (2005j:18010)

[Wei99] M. Weiss, *Embeddings from the point of view of immersion theory. I*, Geom. Topol. **3** (1999), 67–101 (electronic), DOI:10.2140/gt.1999.3.67. MR1694812 (2000c:57055a)

[Wil14] T. Willwacher, *Stable cohomology of polyvector fields*, Math. Res. Lett. **21** (2014), no. 6, 1501–1530, DOI:10.4310/MRL.2014.v21.n6.a16. MR3335859

[Wil15a] T. Willwacher, *The oriented graph complexes*, Comm. Math. Phys. **334** (2015), no. 3, 1649–1666, DOI:10.1007/s00220-014-2168-9. MR3312446

[Wil15b] T. Willwacher, *Deformation quantization and the Gerstenhaber structure on the homology of knot spaces*. preprint (2015), available at arXiv:1506.07078 [math.QA].