NONCOMMUTATIVE DIFFERENTIAL CALCULUS

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ABSTRACT. This is a note from B. Tsygan's lecture series which was part of masterclass "Algebraic structure of Hochschild complexes" at the University of Copenhagen in October 2015.

From the course description: I will review the current state of noncommutative differential calculus. The term stands for the theory that generalizes classical algebraic structures arising in differential calculus on manifolds to make them valid for any associative algebra (or, more generally, any differential graded category) instead of the algebra of functions on a manifold. The role of differential forms and multi-vector fields in this new theory is played by the Hochschild complexes of our algebra. The generalized algebraic structures from classical calculus are provided by the action of various operads on these complexes. I will summarize the current state of the subject as developed in the works of Kontsevich and Soibelman, Tamarkin, Willwacher, and other authors, as well as my own works in collaboration with Dolgushev, Nest, and Tamarkin.

1. HOCHSCHILD AND CYCLIC HOMOLOGIES

1.1. **Basic definitions.** Throughout the course k denotes a field of characteristic 0. Let A be a unital associative algebra over k. We put $\bar{A} := A/k \cdot 1$ and

$$\mathrm{C}_n(A) := \mathrm{C}_n(A, A) = A \otimes \bar{A}^{\otimes n}$$

for each integer $n \geq 0$. Define $b \colon \mathrm{C}_n(A) \to \mathrm{C}_{n-1}(A)$ and $B \colon \mathrm{C}_n(A) \to \mathrm{C}_{n+1}(A)$ by

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots a_n + (-1)^n a_n a_0 \otimes a_1 \ldots \otimes a_{n-1},$$

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{j-1}.$$

Note that we have $bB + Bb = b^2 = B^2 = 0$.

Definition 1.1. Let u be a formal variable of degree $|u| = -2$. We consider the following complexes:

(reduced) Hochschild complex: $(\mathrm{C}_\bullet(A), b)$,
negative cyclic complex: $\mathrm{CC}_\bullet^-(A) := (\mathrm{C}_\bullet(A)[\![u]\!], b + uB)$,
periodic cyclic complex: $\mathrm{CC}_\bullet^{\mathrm{per}}(A) := (\mathrm{C}_\bullet(A)(\!(u)\!), b + uB)$,
cyclic complex: $\mathrm{CC}_\bullet(A) := (\mathrm{C}_\bullet(A)(\!(u)\!)/u\mathrm{C}_\bullet(A)[\![u]\!], b + uB)$.

Their homology groups are respectively denoted by $\mathrm{HH}_\bullet(A)$, $\mathrm{HC}_\bullet^-(A)$, $\mathrm{HC}_\bullet^{\mathrm{per}}(A)$, and $\mathrm{HC}_\bullet(A)$. Here, $\mathrm{C}_\bullet(A)[\![u]\!]$ is the space of formal power series in the formal variable u with coefficients in $\mathrm{C}_\bullet(A)$. More formally, it is just the infinite direct product $\mathrm{C}_\bullet(A)^\infty$, where $(x_0, x_1, \cdots)$ for $x_n \in \mathrm{C}_\bullet(A)$ corresponds to the series $\sum_{n=0}^\infty x_n u^n$. Similarly, $\mathrm{C}_\bullet(\!(u)\!) = \mathrm{C}_\bullet[\![u]\!][u^{-1}]$ denotes the space of Laurent power series $\sum_{n=-\infty}^\infty x_n u^n$ with $x_n = 0$ for $n \ll 0$.

The diagram of Figure 1 clears up the above definitions. Here, the Hochschild complex is given by the column marked $(*)$, the negative cyclic complex is obtained by considering this column and the ones to the right of it, the cyclic complex is obtained by removing the columns to the right of $(*)$, and the total double complex gives us the periodic cyclic complex.

Remark 1.2. It is sometimes convenient to work with the *unreduced Hochschild complex* $\mathrm{C}_\bullet^{\mathrm{full}}(A) = A^{\otimes \bullet + 1}$ with the Hochschild differential b given by the same formula as above. Note that the obvious surjection $\mathrm{C}_\bullet^{\mathrm{full}}(A) \to \mathrm{C}_\bullet(A)$ admits a homotopy inverse coming from insertion of unit, and the operator B lifts to a differential on $\mathrm{C}_\bullet^{\mathrm{full}}(A)$.

$$\text{FIGURE 1. periodic cyclic complex}$$

1.2. **Operations.** In general, we can consider the compositions of the following basic operations:
- cyclic permutation $a_0 \otimes \cdots \otimes a_n \mapsto a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$,
- taking product: $a_0 \otimes \cdots \otimes a_n \mapsto a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$,
- insertion of unit: $a_0 \otimes \cdots \otimes a_n \mapsto a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n$.

The operators b and B are examples of operations on $C_\bullet^{\text{full}}(A)$. More generally, an operation would look like

$$a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_j a_{j+1} a_{j+2} \otimes 1 \otimes a_{j+3} \otimes \ldots \otimes 1 \otimes a_{n-1} \otimes a_n a_0 \otimes a_1 a_2 \otimes \ldots \otimes a_{j-2} a_{j-1}.$$

Let us denote the sets of the operations as above by

$$\Lambda(n,m) := \left\{ \text{``operation''} A^{\otimes n+1} \to A^{\otimes m+1} \right\}.$$

We obtain a category with objects $0, 1, 2, \ldots$ and morphism sets $\Lambda(n,m)$, which is Connes's cyclic category Λ.

Theorem 1.3 ([Con83])**.** *The category Λ is equivalent to its opposite Λ^{op}.*

Proof. Suppose that A has a trace $\text{Tr} \colon A/[A,A] \to k$, and consider the pairing

$$\langle a_0 \otimes \ldots \otimes a_n, b_0 \otimes \ldots \otimes b_n \rangle := \text{Tr}(a_0 b_0 \ldots a_n b_n).$$

When ϕ is in $\Lambda(n,m)$, we can find a $\phi^* \in \Lambda(m,n)$ such that

$$\langle \phi(a_0 \otimes \ldots \otimes a_n), b_0 \otimes \ldots \otimes b_m \rangle = \langle a_0 \otimes \ldots \otimes a_n, \phi^*(b_0 \otimes \ldots \otimes b_m) \rangle$$

by an "universal procedure" which makes sense independent of Tr. This defines a contravariant functor from Λ to itself. Moreover this is involutive, that is $\phi^{**} = \phi$, which shows the assertion. $\square$

Example 1.4. Suppose $\phi(a_0 \otimes \ldots \otimes a_3) = a_2 a_3 \otimes 1 \otimes a_0 a_1 \otimes 1$. Then we have

$$\langle \phi(a_0 \otimes \ldots \otimes a_3), b_0 \otimes \ldots \otimes b_3 \rangle = \text{Tr}(a_2 a_3 b_0 b_1 a_0 a_1 b_2 b_3) = \text{Tr}(a_0 a_1 b_2 b_3 a_2 a_3 b_0 b_1),$$

by traciality. From this we find that $\phi^*(b_0 \otimes \ldots \otimes b_3) = 1 \otimes b_2 b_3 \otimes 1 \otimes b_0 b_1$.

1.3. **Hochschild chains and forms.**

Definition 1.5 (Noncommutative forms)**.** Let $\Omega^\bullet(A)$ denote the algebra generated by the symbols a and da for $a \in A$ subject to the following relations.

- da is k-linear in a, that is, $d(\lambda a + \mu b) = \lambda da + \mu db$,
- $d1 = 0$,
- the Leibniz rule $d(ab) = adb + (da)b$ holds, and
- the symbols $a \in A$ satisfy the relations in the algebra A.

Using the Leibniz rule, any element of $\Omega^\bullet(A)$ can be presented as a linear combination of $a_0 da_1 \cdots da_n$. We endow $\Omega^\bullet(A)$ with the grading by $|a_0 da_1 \ldots da_n| = n$, and the differential d characterized by

$$d \colon a \mapsto da \mapsto 0, \quad d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 d\omega_2$$

for $\omega_1, \omega_2 \in \Omega^\bullet(A)$ and ω_1 homogeneous. This satisfies $d^2 = 0$, hence $\Omega^\bullet(A)$ is a differential graded algebra. Note also that

$$\Omega^n(A) = A(dA)^n \simeq A \otimes \bar{A}^{\otimes n} = \mathrm{C}_n(A),$$

which implies $\Omega^\bullet(A) \simeq \mathrm{C}_\bullet(A)$. Under this isomorphism we find that d is something like B, in the sense that

$$a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_n$$

corresponds to

$$d \colon a_0 da_1 \ldots da_n \mapsto da_0 da_1 \ldots da_n.$$

In fact, there is also an analog ι_Δ (Ginzburg–Schedler [GS12a]) of b on the $\Omega^\bullet(A)$ side as follows. Imagine that we have a trace Tr again, and consider another pairing

$$\langle a_0 \otimes \ldots \otimes a_n, b_0 \otimes \ldots \otimes b_n \rangle = \mathrm{Tr}\,(b_0 a_0 [a_1, b_1] \ldots [a_n, b_n])$$

Then we define ι_Δ to be the dual of B, i.e., the formula

$$\langle a_0 \otimes \ldots \otimes a_{n+1}, B(b_0 \otimes \ldots \otimes b_n) \rangle = \langle i_\Delta(a_0 \otimes \ldots \otimes a_{n+1}), b_0 \otimes \ldots \otimes b_n \rangle$$

defines ι_Δ. Note that similarly we have

$$\langle a_0 \otimes \ldots \otimes a_n, b(b_0 \otimes \ldots \otimes b_{n+1}) \rangle = \langle d(a_0 \otimes \ldots \otimes a_n), b_0 \otimes \ldots \otimes b_{n+1} \rangle.$$

Proposition 1.6. *The "Hochschild–Kostant–Rosenberg" map*

$$\phi_{\mathrm{HKR}}^{\mathrm{nc}} \colon \mathrm{C}_\bullet(A) \to \Omega^\bullet(A), \quad a_0 \otimes \ldots \otimes a_n \mapsto \frac{1}{(n+1)!} \sum_{j=0}^n (-1)^{(n-j)j}(da_{j+1} \ldots da_n)a_0 da_1 \ldots da_j$$

intertwines b with ι_Δ and B with d.

We thus obtain a map of complexes

$$\phi_{\mathrm{HKR}}^{\mathrm{nc}} \colon (\mathrm{C}_\bullet(A)[\![u]\!], b + uB) \to (\Omega^\bullet(A)[\![u]\!], \iota_\Delta + ud).$$

However, one should be aware that ϕ_{HKR} is *not* an isomorphism on homology.

Theorem 1.7 ([GS12a])**.** *The map $\phi_{\mathrm{HKR}}^{\mathrm{nc}}$ becomes a quasi-isomorphism after inverting u, i.e., after passing to the periodic cyclic complex.*

Remark 1.8. When A is commutative, imposing $dadb = -dbda$, we obtain the space of Kähler forms $\Omega_{A/k}^\bullet$. The composition of the quotient map

$$(\Omega^\bullet(A)[\![u]\!], \iota_\Delta + ud) \to (\Omega_{A/k}^\bullet[\![u]\!], ud_{\mathrm{dR}})$$

with $\phi_{\mathrm{HKR}}^{\mathrm{nc}}$ is a quasi-isomorphism when A is regular (smooth Nötherian, or projective limits of such).

Theorem 1.9 ([GS12a])**.** *The map ι_Δ descends to a linear map*

$$\Omega^\bullet(A) \Big/ [\Omega^\bullet(A), \Omega^\bullet(A)] \to \Omega^{\bullet+1}(A).$$

Its kernel is isomorphic to the Hochschild homology of A:

$$\mathrm{Ker}\left(\Omega^\bullet(A) \Big/ [\Omega^\bullet(A), \Omega^\bullet(A)] \to \Omega^{\bullet+1}(A) \right) \simeq \mathrm{HH}_\bullet(A).$$

2. CURVED STRUCTURES

2.1. Curved differential graded algebras and modules.
The following notion adds an analogue of curvature to differential graded algebras.

Definition 2.1 ([GJ90, Pos93]). A (nonunital) *curved differential graded algebra* (*curved dga* for short) is a triple $(A^\bullet, D, R)$, where

- $A^\bullet = \bigoplus_n A^n$ is a graded algebra.
- $D \colon A^\bullet \to A^{\bullet+1}$ is a linear map satisfying $D(ab) = (Da) \cdot b + (-1)^{|a|} a \cdot Db$.
- R is an element of A^2 satisfying $D^2(x) = \mathrm{ad}(R)(x) = [R, x]$ and $D(R) = 0$.

Note that $D^2 = \mathrm{ad}(R)$ alone implies $\mathrm{ad}(D(R)) = 0$, since one has $[D, D^2] = 0$.

Definition 2.2. A curved morphism $(A^\bullet, D_A, R_A) \to (B^\bullet, D_B, R_B)$ is a pair (β, F), where $F \colon A^\bullet \to B^\bullet$ is a morphism of graded algebras ($|F| = 0$), and β is an element of B^1 such that
$$[F, D] := F \circ D_A - D_B \circ F = \mathrm{ad}(\beta) \circ F, \quad F(R_A) - R_B = D_B\beta + \beta^2$$
holds. Again looking at $[F, D^2]$, the first condition already implies that $F(R_A) - R_B - (D_B\beta + \beta^2)$ commutes with the image of F.

Example 2.3. Suppose that F is invertible. Then the above means that
$$F D_A F^{-1} = D_B + \mathrm{ad}(\beta).$$

Definition 2.4. A *curved module* over a curved dga $(A^\bullet, D_A, R)$ is a pair $(V^\bullet, D_V)$, where

- $V^\bullet$ is a graded $A^\bullet$-module,
- $D_V \colon V^\bullet \to V^{\bullet+1}$ a linear map satisfying $D_V(av) = (D_A a)v + (-1)^{|a|} a D_V v$ for all $a \in A$ and $v \in V^\bullet$,
- $D_V^2 v = Rv$.

2.2. Curved differential graded categories and modules.
Let us give a categorified notion of the curved dg algebras and modules.

Definition 2.5. A *curved differential graded category* $\mathcal{A}^\bullet$ is given by the following data:

- a set of objects $\mathcal{X} = \mathrm{ob}(\mathcal{A}^\bullet) \ni X, Y, \ldots$,
- a graded vector space $\mathcal{A}^\bullet(X, Y)$ for each $X, Y \in \mathcal{X}$,
- an associative linear map $\mathcal{A}^\bullet(X, Y) \otimes \mathcal{A}^\bullet(Y, Z) \to \mathcal{A}^\bullet(X, Z)$ of degree 0,
- an element $1_X \in \mathcal{A}^0(X, X)$ for all $X \in \mathcal{X}$, which is a unit for the above product map,
- a linear map $D_{X,Y} \colon \mathcal{A}^\bullet(X, Y) \to \mathcal{A}^{\bullet+1}(X, Y)$ for all $X, Y \in \mathcal{X}$,
- an element $R_X \in \mathcal{A}^2(X, X)$ for all $X \in \mathcal{X}$,

such that
$$D(a_1 a_2) = (D a_1)a_2 + (-1)^{)}|a_1|a_1 D a_2, \quad D^2 a = R_X a - a R_Y$$
holds for for all $a, a_1 \in \mathcal{A}^\bullet(X, Y)$, $a_2 \in \mathcal{A}^\bullet(Y, Z)$ and $X, Y, Z \in \mathcal{X}$.

Definition 2.6. A *curved differential graded module* over a curved dg category $\mathcal{A}^\bullet$ is given by the following data:

- a family of graded vector spaces $V^\bullet(X)$ for $X \in \mathcal{X}$,
- a family of linear maps $\mathcal{A}^\bullet(X, Y) \otimes V^\bullet(Y) \to V^\bullet(X)$ for $X, Y \in \mathcal{X}$,
- a family of linear maps $D_{V(X)} \colon V^\bullet(X) \to V^{\bullet+1}(X)$ for $X \in \mathcal{X}$,

such that

- $(a_1 a_2)v = a_1(a_2 v)$ for all $a_1 \in \mathcal{A}^\bullet(X, Y)$, $a_2 \in \mathcal{A}^\bullet(Y, Z)$, $v \in V^\bullet(Z)$ and all $X, Y, Z \in \mathcal{X}$
- $D_{V(X)}(av) = (Da)v + (-1)^{|a|} a D_{V(Y)} v$ for all $a \in \mathcal{A}^\bullet(X, Y)$, $v \in V^\bullet(Y)$ and $X, Y \in \mathcal{X}$
- $D_{V(X)}^2 = R_X v$ for all $v \in V^\bullet(X)$ for all $X \in \mathcal{X}$.

Definition 2.7. Let $(\mathcal{A}, D_\mathcal{A}, R_\mathcal{A})$ and $(\mathcal{B}, D_\mathcal{B}, R_\mathcal{B})$ be curved dg categories with object sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. A *curved differential graded functor* from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

- a map $F \colon \mathcal{X} \to \mathcal{Y}$

- a linear map $F\colon \mathcal{A}^\bullet(X,Y) \to \mathcal{B}^\bullet(FX, FY)$ for all $X, Y \in \mathcal{X}$
- elements $\beta_X \in \mathcal{B}^1(FX, FX)$ for each $X \in \mathcal{X}$.

such that

- $F(a_1 a_2) = F(a_1)F(a_2)$ for all $a_1 \in \mathcal{A}^\bullet(X,Y)$, $a_2 \in \mathcal{A}^\bullet(Y,Z)$ and all $X, Y, Z \in \mathcal{X}$
- $(FD_\mathcal{A} - D_\mathcal{B}F)(a) = \beta_X F(a) \mp F(a)\beta_Y$ for all $a \in \mathcal{A}^\bullet(X,Y)$ and all $X, Y \in \mathcal{X}$
- $F(R_{\mathcal{A},X}) - R_{\mathcal{B},FX} =$ "exercise"

Example 2.8 (curved dg category). Denote by $\mathrm{Premod}(k)$ the dg category with:

- objects are the pairs $(V^\bullet, D)$, where $V^\bullet$ is a graded vector space and D is a degree 1 endomorphism of $V^\bullet \to$,
- morphisms are given by $\mathcal{A}^\bullet((V_1^\bullet, D_1), (V_2^\bullet, D_2)) = \mathrm{Hom}^\bullet(V_2, V_1)$,
- $d\varphi = D_1 \circ \varphi \mp \varphi \circ D_2$ is the derivation $\mathcal{A}^\bullet((V_1^\bullet, D_1), (V_2^\bullet, D_2)) \to \mathcal{A}^{\bullet+1}((V_1^\bullet, D_1), (V_2^\bullet, D_2))$, and
- $R_{(V,D)}$ is given by $D^2 \in \mathrm{Hom}^2(V^\bullet, V^\bullet)$ for all $(V^\bullet, D)$.

Exercise 2.9. Show that a curved dg module over a curved dg category is given by a curved dg functor

$$F\colon \mathcal{A}^\bullet \to \mathrm{Premod}(k),$$

where the $\beta_X \in \mathcal{B}^1(FX, FX)$ are given by the D_{FX}.

3. Hochschild and cyclic homology for curved dg categories

Next we are going to define the Hochschild and cyclic homology for curved dg categories. Let us fix a curved dg category $(\mathcal{A}^\bullet, D, R)$.

Definition 3.1. We put

$$\mathrm{C}_\bullet(\mathcal{A}^\bullet) := \bigoplus_{\substack{n \geq 0 \\ X_0, \ldots, X_n \in \mathcal{X}}} \mathcal{A}^\bullet(X_0, X_1) \otimes \bar{\mathcal{A}}^\bullet(X_1, X_2) \otimes \ldots \otimes \bar{\mathcal{A}}^\bullet(X_n, X_0),$$

where

$$\bar{\mathcal{A}}^\bullet(X,Y) := \begin{cases} \mathcal{A}^\bullet(X,Y) & (X \neq Y) \\ \mathcal{A}^\bullet(X,X)\big/ k \cdot 1_X & (X = Y). \end{cases}$$

We have the following operations on $\mathrm{C}_\bullet(\mathcal{A}^\bullet)$:

$$|a_0 \otimes a_1 \otimes \ldots \otimes a_n| = n - \sum_{i=0}^n |a_i|,$$

$$b(a_0 \otimes \ldots \otimes a_n) = \left(\sum_{j=0}^{n-1} \pm a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n \right) \pm a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1},$$

$$d(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^n \pm a_0 \otimes \ldots \otimes Da_j \otimes \ldots \otimes a_n$$

$$\mathcal{L}_R(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^n \pm a_0 \otimes \ldots \otimes a_j \otimes R_{x_{j+1}} \otimes a_{j+1} \otimes \ldots \otimes a_n.$$

The signs are determined by the following rules. When one reads from the left to the right on the expression $a_0 \otimes a_1 \otimes \ldots$,

- passing an a_i gives a factor of $(-1)^{|a_i|}$,
- passing an $\otimes$ give a factor of -1.

Similarly, to bring a_n from the right to the left, we will have the factors $(-1)^{|a_n||a_i|}$ from the a_i and $(-1)^{|a_n||\otimes|}$ from the $\otimes$'s.

Now put $\mathbf{b} := b + d + \mathcal{L}_R$, and note that $\mathbf{b}^2 = 0$. We also consider

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} \pm 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_j$$

where we treat "$a_j \otimes$" as an object of degree $|a_i| + 1$.

3.1. Chern character.
Suppose that $\mathcal{A}^{\bullet}$ is an exact category, so that the algebraic K-groups $K_j(\mathcal{A})$ for $j = 0, 1, \ldots$ make sense through Quillen's Q-construction. We want to define the "Chern character" map from these groups to the negative cyclic cohomology groups of $\mathcal{A}^{\bullet}$.

3.1.1. Flat case.
Suppose that $R = 0$. Then for $X \in \mathcal{X}$ we see that

$$\mathrm{Ch}(X) = 1_X \in \mathrm{C}_0(\mathcal{A}) = \mathcal{A}^{\bullet}(X, X) \otimes \bar{\mathcal{A}}^{\otimes 0}$$

defines a $(\mathbf{b}, B)$-cycle, since

$$\mathrm{Ch}(X) \in \mathrm{Ker}\, b \cap \mathrm{Ker}\, B \cap \mathrm{Ker}\, d.$$

So we find the map $\mathrm{Ch} \colon K_0(\mathcal{A}) \to \mathrm{HC}_0^{-}(\mathcal{A})$.

3.1.2. Curved case.
Let us treat the general case $R \neq 0$. Put $\mathrm{CC}_{\bullet}^{-}(\mathcal{A}) := (\mathrm{C}_{\bullet}(\mathcal{A})[\![u]\!], \mathbf{b} + uB)$. We want to find maps

$$\mathrm{Ch} \colon K_j(\mathcal{A}) \to \mathrm{HC}_j^{-}(\mathcal{A}),$$

which would be analogue of the Chern(–Connes–Karoubi) character maps [Con85, Kar87] for algebras. In particular, for $j = 0$ we want to find $\mathrm{Ch}(X) \in \mathrm{CC}_0^{-}(\mathcal{A})$ such that $(\mathbf{b} + uB)\,\mathrm{Ch}(X) = 0$.

How do we find such $\mathrm{Ch}(X)$? Chasing the diagram, we find

$$
\begin{array}{ccc}
1_X & \xrightarrow{\ L_R\ } & 1_X \otimes R_X \\[2mm]
 & & \Big\uparrow{\scriptstyle uB} \\[2mm]
u^{-1}R_X & \xrightarrow{\ \ \mathbf{b}\ \ } & u^{-1}R_X \otimes R_X \\[2mm]
 & & {\scriptstyle uB}\Big\uparrow \\[2mm]
 & & \text{?something?}
\end{array}
$$

So the ansatz is $\mathrm{Ch}(X) = 1_X + u^{-1}R_X +$ something. Since the terms in the above diagram only contain R_X, let us look at the commutative ring $k[R]$. Then we have the Hochschild–Kostant–Rosenberg quasi-isomorphism

$$\mathrm{CC}_{\bullet}^{\mathrm{per}}(k[R]) \to (\Omega_{k[R]/k}^{\bullet}((u)), u d_{\mathrm{dR}} + dR \wedge).$$

Here, $\mathrm{CC}_{\bullet}^{\mathrm{per}}(k[R])$ has the differential $b + uB + \text{``}\mathcal{L}_R\text{''}$, corresponding to the $\mathbf{b}$ operator. We see that the naive solution to $(ud + dR\wedge)x = 0$ in $\Omega_{k[R]/k}^{\bullet}((u))$ is $x = e^{-\frac{R}{u}}$. So we are led to the formula

$$\mathrm{Ch}(X) = e^{-\frac{R_X}{u}} + O,$$

where O is something in the kernel of the HKR map. Note that the above expression actually makes sense when R is nilpotent.

4. Operations on Hochschild/cyclic complexes

4.1. Hochschild cochains.
Let $A^{\bullet}$ be a graded vector space, and let us put

$$\mathrm{C}^{*}(A^{\bullet}, A^{\bullet}) = \prod_{\substack{n+m=*,\\ n \geq 0}} \mathrm{Hom}^{m}(A^{\otimes n}, A).$$

Following the case of associative algebras, we call this the space of Hochschild cochains. We can consider the *brace operation* (due at least to Gerstenhaber): given two cochains $\varphi \colon (A^{\bullet})^{\otimes n} \to A$ and $\psi \colon (A^{\bullet})^{\otimes m} \to A$, one obtains $\varphi\{\psi\} \colon (A^{\bullet})^{\otimes n+m-1} \to A$ defined by

$$\varphi\{\psi\}(a_1, \ldots, a_{n+m-1}) = \sum_{j} \pm \varphi(a_1, \ldots, a_j, \psi(a_{j+1}, \ldots, a_{j+m}), \ldots, a_{n+m-1}).$$

This induces the *Gerstenhaber bracket* of Hochschild cochains,

$$[\varphi, \psi]_G = \varphi\{\psi\} \pm \psi\{\varphi\}.$$

For example, when $A^\bullet = A^0$, we have $[\varphi, \psi]_G = \varphi \circ \psi - \psi \circ \varphi$ for $\varphi, \psi \in \mathrm{C}^1(A, A)$.

Theorem 4.1 ([Ger63])**.** *The bracket $[\cdot, \cdot]_G$ is a graded Lie bracket on $\mathrm{C}^{*+1}(A^\bullet, A^\bullet)$.*

Let $m = m_0 + m_1 + m_2 + \ldots$ with $m_i \in \mathrm{Hom}^{2-n}\left((A^\bullet)^{\otimes n}, A^\bullet\right)$ be a cochain of $*$-degree 2. In other words, we have

$$m_0 \in A^2, \quad m_1 \colon A^\bullet \to A^{\bullet+1}, \quad m_2 \colon A^\bullet \otimes A^\bullet \to A^\bullet, \quad m_3 \colon (A^\bullet)^{\otimes 3} \to A^{\bullet-1}$$

and so on. Then, if $m_i = 0$ for $i \geq 3$ and $m\{m\} = 0$, we have a structure nonunital curved dg algebra on A with $R = m_0$, $D = m_1$, and $xy = m_2(x, y)$. Thus, without restriction on the vanishing of m_i, we can say that $m\{m\} = 0$ defines a structure of curved A_∞ algebra. Note that m has degree 1 in the graded Lie algebra $C^{*+1}(A^\bullet, A^\bullet)$, hence this condition can be also expressed as $[m, m]_G = 0$.

We can generalize the above construction to obtain more operations by putting

$$\varphi\{\psi_1, \ldots, \psi_m\}(a_1, \ldots, a_N) =$$
$$\sum_{j_i} \pm \varphi(a_1, \ldots, a_{j_1}, \psi_1(a_{j_1+1}, \ldots, a_{j_2}), a_{j_2+1}, \ldots, \psi_m(a_{j_{2m-1}+1}, \ldots, a_{j_{2m}}), a_{j_{2m}+1}, \ldots, a_N)$$

for $\varphi, \psi_1, \ldots, \psi_m \in \mathrm{C}^*(A^\bullet, A^\bullet)$, which yield maps

$$\mathrm{C}^* \otimes (\mathrm{C}^*)^{\otimes m} \to \mathrm{C}^*$$

satisfying some interesting algebraic relations called a *brace algebra* [GV95].

4.2. **Categorifying brace operations.** Note that we can also consider the cochain space for categories as

$$\mathrm{C}^*(\mathcal{A}^\bullet, \mathcal{A}^\bullet) = \prod_{\substack{n \geq 0, \\ x_0, \ldots, x_n, \\ m+n=*}} \mathrm{Hom}^m\left(\mathcal{A}^\bullet(x_0, x_1) \otimes \ldots \otimes \mathcal{A}^\bullet(x_{n-1}, x_n), \mathcal{A}^\bullet(x_0, x_n)\right)$$

for a category $\mathcal{A}^\bullet$ enriched by graded vector spaces. What does the brace algebra structure on this space capture? The answer to this question can be understood with insight from [Tam07].

The underlying fundamental scheme is the formalism of 2-*categories*. Recall that usual categories form a 2-category as follows:

- 0-*cells* are categories $\mathcal{C}, \mathcal{D}, \ldots$,
- 1-*morphisms/1-cells* between the 0-cells $\mathcal{C}$ and $\mathcal{D}$ are functors $F \colon \mathcal{C} \to \mathcal{D}$,
- 2-*morphisms/2-cells* between the 1-morphisms $F, G \colon \mathcal{C} \to \mathcal{D}$ are natural transformations $\eta \colon F \to G$.

The analogue of this for dg categories was clarified in an article in [Tam07]. In our context, we should consider the following "pre curved dg-2-category":

OBJECTS (0-cells): $A, B, \ldots$ algebras (or more generally dg categories, A_∞ categories...),
MORPHISMS: a curved dg category $\mathcal{C}^\bullet(A, B)$, given by
    objects (1-morphisms): linear maps $f \colon A \to B$.
    morphisms (2-morphisms): $\mathcal{C}^\bullet(f, g) = \prod_{n \geq 0} \mathrm{Hom}(A^\otimes, B)$, with the differential $_f\delta_g$ given

    by

$$_f\delta_g\, \varphi(a_1, \ldots, a_{n+1}) = f(a_1)\varphi(a_2, \ldots, a_{n+1}) + \sum \pm\varphi(a_1, \ldots, a_j a_{j+1} \ldots, a_{n+1})$$
$$\pm \varphi(a_1, \ldots, a_n)g(a_{n+1}).$$

The composition $\mathcal{C}^\bullet(f, g) \otimes \mathcal{C}^\bullet(g, h) \to \mathcal{C}^\bullet(f, h)$ is given by the *cup product*

$$\varphi \cup \psi(a_1, \ldots, a_{n+m}) = \pm\varphi(a_1, \ldots, a_n)\psi(a_{n+1}, \ldots, a_{n+m}).$$

One immediate consequence of the above definitions is the following Leibniz rule:

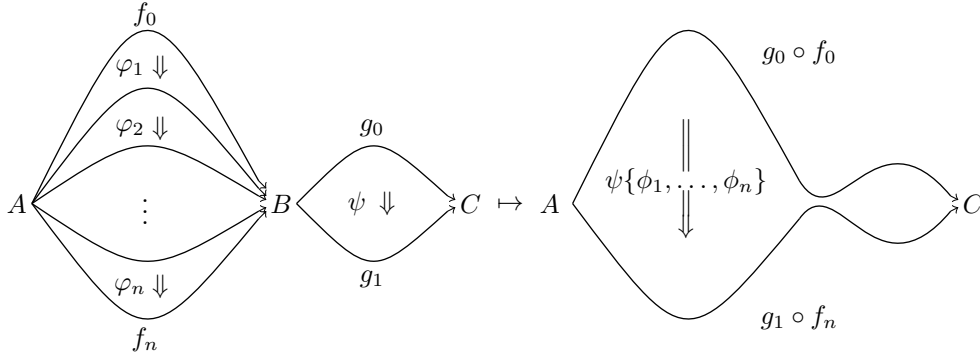$$_f\delta_h(\varphi \cup \psi) = (\,_f\delta_g\,\varphi) \cup \psi \pm \varphi \cup (\,_g\delta_h\,\psi).$$

Note also that $(\,_f\delta_g)^2\,\varphi = \rho_f\varphi \cup \varphi - \varphi \cup \rho_g$ holds for the "Quillen curvature form"

$$\rho_f(a_1, a_2) = f(a_1)f(a_2) - f(a_1 a_2)$$

in $\mathcal{C}^2(A, B)$. Observe that $_f\delta_f\,\rho_f = 0$.

Exercise 4.2. Show that if $A = B$ and $f = g = \mathrm{Id}_A$, then $_f\delta_g = [m_2, \cdot]_G = \delta$.

Let $f_i \colon A \to B$ $(i = 0, 1, \ldots, n)$ and $g_i \colon B \to C$ $(i = 0, 1)$ be linear maps, and consider the 'cochains' $\phi_i \in \mathcal{C}^\bullet(A, B)(f_{i-1}, f_i)$ and $\psi \in \mathcal{C}^\bullet(B, C)(g_0, g_1)$. Then we can define the "brace" operation

by the formula

$$\psi\{\varphi_1, \ldots, \varphi_n\}(a_1, a_2, \ldots, a_N) = \sum \pm\psi(f_0(a_1), \ldots, f_0(a_{i_1}), \varphi_1(a_{i_1+1}, \ldots, a_{i_2}), f_1(a_{i_2+1}), \ldots,$$
$$f_1(a_{i_3}), \varphi_2(a_{i_3+1}, \ldots, a_{i_4}), \ldots, \varphi_n(a_{i_{2n-1}+1}, \ldots, a_{i_{2n}}), f_n(a_{i_{2n}+1}), \ldots, f_n(a_N)).$$

The corresponding "Steenrod formula" is

$$_{g_0 \circ f_0}\delta_{g_1 \circ f_1}(\psi\{\varphi\}) - (_{g_0}\delta_{g_1}\psi)\{\varphi\} + \psi\{_{f_0}\delta_{f_1}\varphi\} = (\psi f_1) \cup (g_0\varphi) \mp (\psi f_0) \cup (g_1\varphi),$$

where $(\psi f_0)(a_1, \ldots)$ is given by $\psi(f_0(a_1), \ldots)$. Thus, given the input as in Figure 2a, the right hand side of the Steenrod formula is represented by Figure 2b.

(A)   (B)
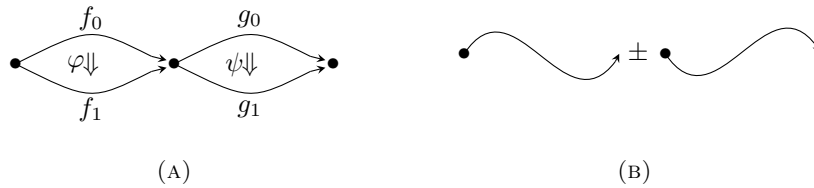
FIGURE 2. Steenrod formula

To make the braces a multilinear operation, we consider the *bar complex*. When $\mathcal{A}^\bullet$ is a dg category, we can consider a new dg cocategory $\mathrm{Bar}(\mathcal{A}^\bullet)$ which has the same objects $X, Y, \ldots$ is $\mathcal{A}$, but

$$(1) \qquad \mathrm{Bar}(\mathcal{A}^\bullet)(X, Y) = \bigoplus_{\substack{n \geq 1, \\ X_1, \ldots, X_{n-1}}} \mathcal{A}^\bullet(X, X_1) \otimes \ldots \otimes \mathcal{A}^\bullet(X_{n-1}, Y),$$

with differential

$$(a_1 | \ldots | a_n) \overset{\partial_{\mathrm{Bar}}}{\longmapsto} \sum \pm(a_1 | \ldots | a_j a_{j+1} | \ldots) + \sum \pm(a_1 | \ldots | da_j | \ldots) + \sum \pm(a_1 | \ldots | a_j | R_{X_j} | \ldots)$$

and cocomposition

$$\mathrm{Bar}(\mathcal{A}^{\bullet})(X, Z) \overset{\Delta_{XYZ}}{\Rightarrow} \mathrm{Bar}(\mathcal{A}^{\bullet})(A, Y) \otimes \mathrm{Bar}\,\mathcal{A}^{\bullet}(Y, Z)$$

given by

$$(a_1 | \dots | a_n) \mapsto \sum_{x_j = y} (a_1 | \dots | a_j) \otimes (a_{j+1} | \dots | a_n).$$

Remark 4.3. Historically the name "bar" refers to the shorthand | for the tensor product in the expression $(a_1 | \dots | a_n)$.

So we get a dg (co)functor

$$\mathrm{Bar}\,\mathcal{C}^{\bullet}(A, B) \otimes \mathrm{Bar}\,\mathcal{C}^{\bullet}(B, C) \to \mathrm{Bar}\,\mathcal{C}^{\bullet}(A, C)$$

where the left hand side is the cocategory with the objects $f \otimes g$ (formal symbols). The functor is given on objects by

$$f \otimes g \mapsto (g \circ f \colon A \to C)$$

and on morphisms by

$$(\varphi_1 | \dots | \varphi_n) \otimes (\psi_1 | \dots | \psi_m) \mapsto \sum \mp (\varphi_1 | \dots | \varphi_{i_1} | \psi_1 \{ \varphi_{i_1+1}, \dots, \varphi_{i_2} \} | \varphi_{i_2+1} | \dots).$$

In terms of pictures, Figure 3a corresponds to $\psi_j \{ \varphi_{i_k}, \dots, \varphi_{i_l} \}$, and Figure 3b corresponds to $g\varphi$. In the end we have constructed a category in cocategories.

(A)　　　　　　　　　　　　(B)
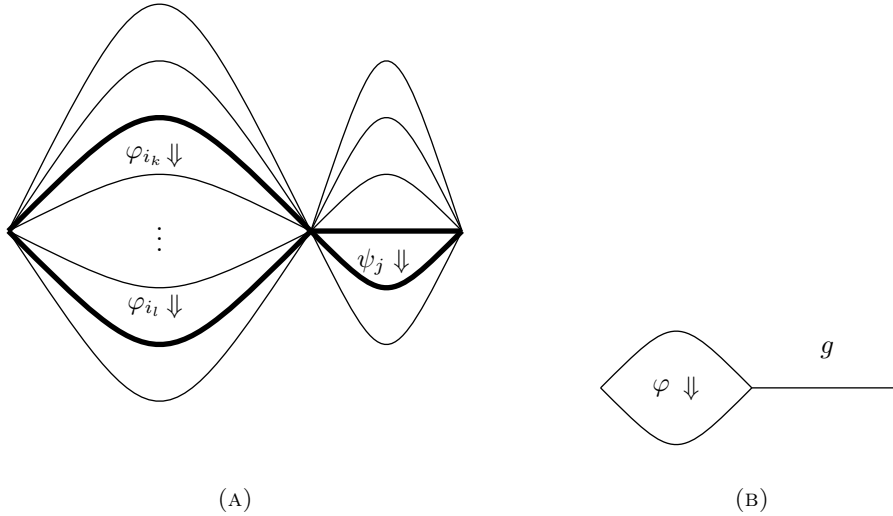
FIGURE 3. brace product

5. A Two-category (in a strictly defined weaker sense)

5.1. **Cobar category.** We should find out the structure of the bar-cobar construction. As before, the case of algebras gives a guiding principle. In that case we have equivalences

$$\mathrm{C}_{\bullet}(\mathrm{Bar}(A)) \simeq \mathrm{C}_{\bullet}(A), \quad \mathrm{CC}_{\bullet}(\mathrm{Bar}(A)) \simeq \mathrm{CC}_{\bullet}^{-}(A).$$

Here, the left hand sides are the Hochschild and cyclic complexes of the bar coalgebra, see for example [Kha97]. Note that $\mathrm{C}_{\bullet}(\mathrm{Bar}(A))$ can be regarded as the bar-cobar construction on A. The reason of these equivalences boils down to the fact that $\mathrm{Bar}(A)$ has cohomological dimension 1, so that the Hochschild complex can be replaced by Quillen's X-complex [Qui88]. Replacing

A with the system $\mathcal{C}^\bullet(A,B)$, we obtain a A_∞-category with morphism sets $\mathrm{CC}_\bullet^-(\mathcal{C}^\bullet(A,B))$, and A_∞-module category given by the $\mathrm{CC}_\bullet^-(A)$. This structure is induced from

$$\mathrm{Bar}(A) \otimes \mathrm{Bar}\,\mathcal{C}^\bullet(A,B) \to \mathrm{Bar}(B).$$

This formalism specializes to the A_∞-algebra $\mathrm{CC}_\bullet^-(\mathrm{C}^\bullet(A,A))$ acting on $\mathrm{CC}_\bullet^-(A)$ [TT05].

More systematically, we can work with the cobar construction at the categorical level. When $\mathcal{D}$ is a cocategory, the morphism space in $\mathrm{Cobar}\,\mathcal{D}$ is defined by the same formula as (1). It has a differential induced by the cocomposition in $\mathcal{D}$, and the composition of morphisms is the concatenation of tensors. We can then take $\mathscr{C}^\bullet(A,B) = \mathrm{Cobar}\,\mathrm{Bar}\,\mathcal{C}^\bullet(A,B)$, which is quasi-isomorphic to $\mathcal{C}^\bullet(A,B)$. Then $\mathscr{C}$ can be regarded as a 2-category, or a category in dg algebras.

Definition 5.1. For coalgebras B_1 and B_2, we denote

$$B_1 \underline{\otimes} B_2 := B_1^+ \otimes B_2^+ \,/ k \otimes k\,.$$

Caution 5.2. Although the shuffle map (Eilenberg–Zilber equivalence)

$$\mathrm{Cobar}\,(B_1 \underline{\otimes} B_2) \to \mathrm{Cobar}(B_1) \otimes \mathrm{Cobar}(B_2),$$

is a morphism of algebras, the map in the other way (Alexander–Whitney map)

$$\mathrm{Cobar}(B_1) \otimes \mathrm{Cobar}(B_2) \to \mathrm{Cobar}\,(B_1 \underline{\otimes} B_2)$$

is not.

Now we are ready to present the higher version of $\mathscr{C}$.

- Given algebras $A_1, A_2, \ldots$, consider the dg category
  $$\mathscr{C}^\bullet(A_1 \to A_2 \to \ldots \to A_n) = \mathrm{Cobar}\,(\mathrm{Bar}\,\mathcal{C}^\bullet(A_1, A_2) \underline{\otimes} \ldots \underline{\otimes} \mathrm{Bar}\,\mathcal{C}^\bullet(A_{n-1}, A_n))\,.$$
- Any way of composing gives rise to a dg functor. For example,
  $$(A_1 \to A_2 \to A_3 \to A_4 \to A_5) \longrightarrow (A_1 \to A_3 \to A_5)$$
  gives
  $$\mathscr{C}^\bullet(A_1 \to \ldots \to A_5) \to \mathscr{C}^\bullet(A_1 \to A_3 \to A_5),$$
  through $\mathrm{Bar}\,(\mathcal{C}^\bullet(A_1, A_2)) \otimes \mathrm{Bar}\,(\mathcal{C}^\bullet(A_2, A_3)) \to \mathrm{Bar}\,(\mathcal{C}^\bullet(A_1, A_3))$. These are *operations of type I*.
- The *operations of type II* are given by the dg functors

(2) $$\mathscr{C}^\bullet(A_1 \to \ldots \to A_n) \xrightarrow{\mathrm{qis}} \mathscr{C}^\bullet(A_1 \to \ldots \to A_m) \otimes \mathscr{C}^\bullet(A_m \to \ldots \to A_n)$$

  for all $1 < m \le n$, induced by the morphism of dg categories
  $$\mathrm{Cobar}(B_1 \underline{\otimes} B_2) \to \mathrm{Cobar}(B_1) \otimes \mathrm{Cobar}(B_2).$$

  Here, the categories on both sides of (2) have the same set of objects. Hence "qis" just means that it induces quasi-isomorphisms on morphism complexes.

The operations of type II have coassociativity, and the two kinds of operations are compatible (cf. Leinster [Lei99]).

Let E_k denote a cofibrant replacement of the chain operad of little disks $\mathrm{C}_{-\bullet}(\mathsf{LD}_k)$. When $A = A_i$ and $A \to A$ is Id_A, we obtain an action of $\mathsf{E}_1 \otimes \mathsf{E}_1$ on $C^*(A, A)$ through the above two kinds of operations. Note that one has weak equivalence[1] $\mathsf{E}_1 \otimes \mathsf{E}_1 \simeq \mathsf{E}_2$ (cf. [Lur14, Section 5.1.2]), i.e.,

$$\mathrm{C}_{-\bullet}\left(\boxed{} \right) \simeq \mathrm{C}_{-\bullet}\left(\boxed{} \right).$$

This explains the the following theorem.

Theorem 5.3 ([Tam98, Hin03]). $\mathrm{C}^\bullet(A, A)$ *has a structure of homotopy Gerstenhaber algebra, whose underlying L_∞ structure is $(\delta, [\cdot, \cdot]_G)$.*

---

[1]The quasi-isomorphism map of operads have the same (essential) ambiguity as the Grothendieck–Teichmüller group [Kon99].

So $(C^\bullet(A, A), \cup, [\cdot, \cdot]_G)$ is a homotopy Gerstenhaber analogue of the Gerstenhaber algebra of polyvector fields $(\bigwedge^\bullet T_X, \wedge, [\cdot, \cdot]_{\mathrm{Sch}})$. Similarly, $C_{-\bullet}(A)$ is an analog of $\Omega_X^{-\bullet}$ [Tsy99].

5.2. 2-category with trace.

Definition 5.4. A 2-*category with a trace* is given by:

> *objects*: $A, B, \ldots$.
> 1-*morphisms*: categories $\mathcal{C}(A, B)$ and functors $\circ \colon \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$.
> 2-*morphisms*: morphisms in $\mathcal{C}(A, B)$.
> *trace*: functors $\mathrm{TR} \colon \mathcal{C}(A, A) \to k\text{-mod}$ endowed with:
>> • natural isomorphism $\mathrm{TR}(M \circ N) \simeq \mathrm{TR}(N \circ M)$ for $M \in \mathcal{C}(A, B)$ and $N \in \mathcal{C}(B, A)$,
>> • natural isomorphism $\mathrm{TR}(M_1 \circ M_2, M_3) \simeq \mathrm{TR}(M_2 \circ M_3, M_1)$ compatible with the above.

This definition is motivated by the following example.

Example 5.5. The k-algebras form a 2-category, by the following convention:

> *objects*: algebras $A, B, \ldots$.
> 1-*morphisms*: bimodules ${}_A M_B$, with composition $M \circ N = M \otimes_B N$.
> 2-*morphisms*: (A, B)-bimodule maps $\varphi \colon M \to N$.

Then, if A is a subalgebra of B (with inclusion denoted by i) and there is another morphism $f \colon A \to B$, we have the bimodule ${}_i B_f$ over A, where $a_1 \cdot b \cdot a_2 = a_1 b f(a_2)$. Note that we have

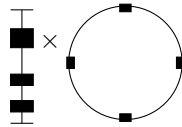$$\mathrm{Ext}_{A \otimes A^{op}}\left({}_i B_f, {}_i B_g\right) \simeq \mathrm{HH}^\bullet\left(A, {}_f B_g\right).$$

The trace functor is given by

$$\mathrm{TR}(M) = M \Big/ [A, M] = \mathrm{HH}_0(A, M),$$

and the natural isomorphism $\mathrm{TR}(M \circ N) \to \mathrm{TR}(N \circ N)$ is given by $[x \otimes y] \mapsto [y \otimes x]$.

The correspondence $A \rightsquigarrow (C^\bullet(A, A), C_{-\bullet}(A))$ can be understood within the framework of 2-category with trace as in Definition 5.4, but in a weaker (Leinster) sense.

The pairs $(C^\bullet(A, A), C_{-\bullet}(A))$ should have a structure of algebra over the 2-colored operad

The first part is LD_1, which encodes the associative/A_∞-algebras. In genera, the algebras over this operad are pairs (A, M), where

> • A is an associative algebra, and
> • M receives an M-valued "trace" map $A^{\otimes n} \to M$.

This gives rise to the algebra $(C^\bullet(A, M), C_{-\bullet}(A, M))$ over the 2-colored operad LDC (little disk cylinder):

$= \mathsf{LDC}(c, c, c, c, c, a; a)$ corresponds to $(C^\bullet)^{\otimes 5} \otimes C_{-\bullet} \to C_{-\bullet}$.

Another example is

$$\mathrm{C}_{-\bullet}\left(\begin{array}{c} \vcenter{\hbox{\includegraphics}} \end{array}\right) = \mathsf{LDC}(c,c,c,c;c) \text{ corresponds to } (\mathrm{C}^\bullet)^{\otimes 4} \to \mathrm{C}^\bullet.$$

The formality theorem could be state as

$$\mathsf{LCD} \xrightarrow{\sim} \mathsf{Calc},$$

where the right hand side is the 2-colored operad which acts on $\left(\bigwedge^\bullet T_X, \Omega_X^{-\bullet}\right)$ [TT00, Tsy04].

Example 5.6 (Homotopy 2-category with trace). Let $f\colon A \to A$ be an algebra endomorphism, so that it fits in the scheme of Example 5.5:

$$A \underset{\mathrm{Id}}{\overset{f}{\leftrightarrows}} A.$$

We set $\mathrm{Tr}(\mathrm{Id}) = \mathrm{C}_\bullet(A, A)$ and $\mathrm{Tr}(f) = \mathrm{C}_\bullet(A, A_f)$. There are two maps from $\mathrm{C}_\bullet(A, A_f)$ to itself, Id and

$$f\colon a_0 \otimes a_1 \otimes \cdots \mapsto f(a_0) \otimes f(a_1) \otimes \cdots.$$

Then $\mathrm{Tr}(f \circ \mathrm{Id}) \simeq \mathrm{Tr}(\mathrm{Id} \circ f)$ indicates that these should be homotopic. Indeed, with

$$S_f\colon \mathrm{C}_\bullet \to \mathrm{C}_{\bullet+1}, \quad a_0 \otimes \ldots \otimes a_n \mapsto \sum \pm 1 \otimes a_j \otimes \ldots \otimes a_0 \otimes f(a_1) \otimes \ldots \otimes f(a_{j-1})$$

we have $[b, S_f] = \mathrm{Id} - f$.

More generally, given algebra homomorphisms

$$A \underset{f}{\overset{g}{\leftrightarrows}} B,$$

we can put

$$_A M_B := {}_{f(A)}A_B, \quad _B N_A := {}_{g(B)}A_A.$$

So $M \otimes_B N = {}_{gf(A)}A_A$ and $N \otimes_A M = {}_{fg(B)}B_B$ and there are equivalences

$$\mathrm{TR}(M \otimes_B N) = \left(C_\bullet\left(A, {}_{gf}A\right), {}_{gf}b\right)$$

$$\Phi_{N,M} \Big\uparrow \Big\downarrow \Phi_{M,N}$$

$$\mathrm{TR}(M \otimes_B N) = \left(C_\bullet\left(B, {}_{fg}A\right), {}_{fg}b\right)$$

(the complexes on the right are usual Hochschild complexes for bimodules) given by

$$\Phi_{M,N}(a_0 \otimes \ldots \otimes a_n) = f(a_0) \otimes \ldots \otimes f(a_n), \quad \Phi_{N,M}(b_0 \otimes \ldots b_n) = g(b_0) \otimes \ldots \otimes g(b_n).$$

Here, the homotopy inverse of $\Phi_{M,N}$ is given by g, and vice versa so that $gf - \mathrm{Id}$ is the boundary of "$_{fg}B$".

5.3. **Curved case.** Recall from last time the definition of the curved dg category $\mathcal{C}^\bullet(A, B)$ (with objects linear maps $A \to B$). Let's try to make sense of

$$\mathrm{CC}_\bullet^-(A) \otimes \mathrm{CC}_\bullet^-(\mathcal{C}^\bullet(A, B)) \to \mathrm{CC}_\bullet^-(B).$$

Then what should be the image of $(a_0 \otimes \ldots \otimes a_n) \otimes \mathrm{Ch}(1_f)$? For any linear map $f \colon A \to B$, one (formally) obtains a map

$$f \colon \mathrm{CC}_\bullet^{\mathrm{per}}(A) \to \mathrm{CC}_\bullet^{\mathrm{per}}(B)$$

which makes sense if

$$R_f(a_1, a_2) := f(a_1 a_2) - f(a_1)f(a_2)$$

has values in a nilpotent ideal $I \subset B$. In this case f is actually defined (cf. Goodwillie's theorem [Goo85]). Now consider the same space with two products, i.e $A_1 = (A, \cdot_1)$ and $A_2 = (A, \cdot_2)$, and the map $f = \mathrm{Id} \colon A \to A$. Then if R_f takes values in a nilpotent ideal (like for deformations) we have that $\mathrm{CC}_\bullet^{\mathrm{per}}(A_1) \simeq \mathrm{CC}_\bullet^{\mathrm{per}}(A_2)$.

5.4. **Conclusion.** Differential graded categories appear in many context

$$
\begin{array}{c}
\text{algebra} \\
\text{geometry} \longrightarrow \text{dg categories,} \\
\text{topology}
\end{array}
$$
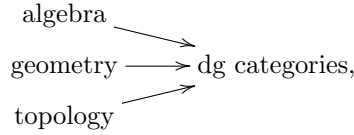
and as Bertland Toën tells us the goal is to get rid of the things of the left column.

6. REPRESENTATION SCHEMES

Let A and B be algebras over k. Let us take a basis $(e_j)_j$ of B, and denote the structure constant by c_{kl}^j, so that $e_k e_l = \sum_j c_{kl}^j e_j$ holds. Then $\mathcal{O}(\mathrm{Rep}(A, B))$ is the commutative k-algebra with

  generators: $\rho^j(a)$ for $a \in A$ and j (linear in a),
  relations: $\rho^j(a_1 a_2) = \sum c_{kl}^j \rho^k(a_1)\rho^l(a_2)$.

We also have the derived version $\mathcal{O}(\mathbb{L}\,\mathrm{Rep}(A, B))$ given by the differential graded algebra $\mathcal{O}(\mathrm{Rep}(\tilde{A}, B))$ for $\tilde{A}$ a cofibrant replacement of A and $d\rho^j(a) := \rho^j(\partial_{\tilde{A}} a)$.

The intuition comes from the case where $B = M_n(k)$ (or a limit of such) [KR00]. In this case GL_n acts on $\mathrm{Rep}(A, B)$. We denote

$$\Omega_{\mathrm{Rep}(A)}^\bullet := \Omega_{\mathcal{O}(\mathrm{Rep}(A,B))/k}^\bullet, \quad \Omega_{\mathrm{GL}_n}^\bullet(\mathrm{Rep}_n(A)) = \mathrm{Hom}(\mathrm{Sym}^* \mathfrak{gl}_n, \Omega_{\mathrm{Rep}(A)}^\bullet)^{\mathrm{GL}_n}.$$

Then one has a natural linear map [GS12b]

$$\mathrm{CC}_\bullet^{\mathrm{per}}(A, A) \longrightarrow \Omega_{\mathrm{GL}_n}^\bullet(\mathrm{Rep}_n(A)),$$

which is a motivation behind the i_Δ differential of [GS12b]. This gives a Kähler style equivariant de Rham complex of the scheme $\mathrm{Rep}_n(A)$ related to $\mathrm{HC}_\bullet^{\mathrm{per}}$ etc. It is still unknown how to get

$$\mathrm{H}_{\mathrm{sing}}^\bullet(E\mathrm{GL}_n \times_{\mathrm{GL}_n} \mathrm{Rep}_n(A))$$

over $\mathbb{C}$.

Consider again the general B (not necessarily a matrix ring). If one uses $\tilde{A} = \mathrm{Cobar}(\mathrm{Bar}(A))$,

$$\mathcal{O}\left(\mathbb{L}\,\mathrm{Rep}(A, B)\right) = \mathrm{C}_{\mathrm{CE}}^\bullet\left(\mathrm{Conv}(\mathrm{Bar}(A), B)\right)$$

where Conv denotes the convolution differential graded Lie algebra. For $B = M_n(k)$, Berest et.al. [BFP$^+$14] showed that

$$\mathrm{H}^\bullet\left(\mathcal{O}\left(\mathbb{L}\,\mathrm{Rep}_n(A)\right)\right) \simeq \mathrm{H}_{\mathrm{CE}}^\bullet\left(\mathfrak{gl}_n(\mathrm{Bar}(A))\right).$$

Now, $\mathrm{C}_{\mathrm{CE}}^\bullet(\mathfrak{gl}_n(\mathrm{Bar}(A)))$ gives rise to $\mathrm{Sym}(\mathrm{CC}^\bullet(\mathrm{Bar}(A)))$ by letting $n \to \infty$, which is "equal" to $\mathrm{CC}^\bullet(A)$.

For commutative A the "classic" understanding is $A = \mathcal{O}(X)$ for $X = \operatorname{Spec} A$ and we study the geometric/topological invariants of X in terms of A, e.g.,

$$H^\bullet_{\text{sing}}(X) = \begin{cases} H^\bullet(\Omega^\bullet_{A/k}, d_{\text{dR}}) & \text{if } A \text{ is smooth,} \\ HC^{\text{per}}_\bullet(A) & \text{in general [Rin63, FT85].} \end{cases}$$

For an non-commutative algebra A we have two approaches:

$$A \qquad \begin{array}{c} \operatorname{Rep} A \\ \\ HC^{\text{per}}_\bullet(A)\text{"} = \text{"}H^\bullet_{\text{sing}}(X) \end{array} \qquad H^\bullet_{\text{sing}}\left(\operatorname{Rep}_n(A) /\!\!/ \operatorname{GL}_n\right),$$

where $\operatorname{Rep} A$ and X should be understood as some hypothetical spaces which are spectra of the non-commutative algebra A, and "$/\!\!/$" stands for the homotopy quotient.

Points of $\mathbb{L}\operatorname{Rep}(A, B)$ are given by morphisms $\operatorname{Spec} k \to \mathbb{L}\operatorname{Rep}(A, B)$, or rather homomorphisms $\mathcal{O}(\mathbb{L}\operatorname{Rep}(A, B)) \to k$ which correspond to the A_∞-morphisms $f \colon A \to B$. Now we have the dictionary

$$\{A_\infty\text{-morphisms} f \colon A \to B\} = \{\text{objects of the dg category } \mathcal{C}^\bullet(A, B)\}.$$

We could hope for a "derived scheme" $\operatorname{Mor}(A, B)$ (a sort of morphism space with object space $\mathbb{L}\operatorname{Rep}(A, B)$) with composition structure in DGA terms.

7. More operations

When A is an e_n-algebra, because of $\mathsf{E}_n \simeq \mathsf{E}_1 \otimes \ldots \otimes \mathsf{E}_1$ (cf. [Lur14, Section 5.1.2]), roughly speaking A has n associative structures. If we take $C_\bullet$ or $CC^-_\bullet$ with respect to one of them, the result should have an e_{n-1}-algebra structure. The motivation here comes from the fact that $C^\bullet = C^\bullet(A, A)$ is a brace algebra, and $CC^-_\bullet(C^\bullet)$ is an A_∞-algebra.

One candidate is the following: if A is an e_n-algebra, $C_\bullet(A)$ becomes an e_{n-1}-algebra by

$$(a_0 \otimes \ldots \otimes a_n) \cdot (b_0 \otimes \ldots \otimes b_m) = (a_0 b_0 \otimes \sum_{j,k} \operatorname{sh}(a_0, \ldots, a_j, b_1, \ldots, b_{k-1}) \otimes \{a_j, b_k\} \otimes \operatorname{sh}(\text{remaining})$$

$$+ \frac{1}{2} \left(\{a_0, b_1\} b_0 \otimes \operatorname{sh}(a_1, \ldots, b_2, \ldots) + \{b_n, a_0\} b_0 \otimes \operatorname{sh}(a_1, \ldots, b_{n-1})\right) + (a \leftrightarrow b).$$

jeg begriper inte det här! (Fröken Bock, *Karlsson på taket*)

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