Recall from the last week

$D \in C^n(A,A) = L(A^{\otimes n}, A)$ defs :

$L_D : C_k(A) \to C_{k-n+1}(A)$, $\sum (-1)^{\circ} (a_0, \dots, D(a_{j+1}, \dots), \dots)$

$+ \sum (-1)^{\circ} (D(a_{k+1}, \dots, a_0, \dots), \dots)$

$b_D : C_k \to C_{k-n}$, $(D(a_{k-n+1}, \dots, a_k) a_0, \dots)$

$B_D : C_k \to C_{k-n+2}$, $\sum (-1)^{\circ} (1, a_{j+1}, \dots, D(a_{k+1}, \dots), \dots, a_0, \dots)$

$\iota_D = b_D - B_D$

Cartan homotopy formula: $[b-B, \iota_D] = L_D - \iota_{\delta(D)}$

$(m_t)_{t \in [0,1]}$ smooth family of assoc alg str. on A

$\gamma_t = m_t - m_0$ $\quad \mathcal{A} = C^\infty([0,1], A)$

$\rightsquigarrow \quad \partial_t + \iota_{\dot\gamma} = L_{\partial_t} - \iota_{\delta(\partial_t)}$ vanishes on $HP_*(\mathcal{A})$

$\partial_t + \iota_{\dot\gamma}$ defs a connection on

$C^\infty([0,1], HP_*(A; m_t))$

$\because \quad [b-B, \partial_t + \iota_{\dot\gamma}] = -\dot b + L_{\dot\gamma}$, but

$[b-B, \partial_t] = -\partial_t b) = L_{\dot\gamma}$

If $(c_t)_t$ is a fam. of chains s.t.

- c_0 is a cycle of $C_*(A; m_0)$
- $(\partial_t + \iota_{\dot\gamma}) c_t = 0$

$\Rightarrow \quad \forall t \quad c_t$ is a cycle of $C_*(A; m_t)$

Q. Can we always solve this? (impossible for chains)

The NC-torus : strategy : $\iota_{\dot\gamma}$ is sq-zero (true on $\Omega^\circ(\mathbb{T}^2)$)

$A = \oplus_{m,n} \mathbb{C} u^m v^n \qquad u^m v^n \cdot u^{m'} v^{n'} = e^{i\theta m' n} u^{m+m'} v^{n+n'}$

$\rightsquigarrow \quad \dot\gamma(u^m v^n, u^{m'} v^{n'}) = i m' n\, e^{i\theta m' n} u^{m+m'} v^{n+n'}$

Notn $\quad \partial_1(u^m v^n) = m u^m v^n, \quad \partial_2(u^m v^n) = n u^m v^n$

$\rightsquigarrow \quad \dot\gamma(a,b) = m_\theta(\partial_2(a), \partial_1(b))$

One trick: by CHF, only the ∂_1-inv chains contribute to HP_*. same for ∂_2

$\because$ $L_D = [b-B, \iota_D]$ for derivation D

On the D-invariant chains ($D = \partial_1$ or ∂_2), ι_D maps cycles to cycles:

$$(b-B)\,\iota_D\, c = L_D c + \iota_D (b-B) c = 0$$

Another trick: work w/ the dual cplx

$$\begin{array}{ccccccc} (C_*(A), b, B) \sim & C_1(A) & \xleftarrow{1-\lambda} & C_1(A) & \xleftarrow{1+\lambda} & C_1 & = CC_*(A) \\ & \downarrow b & & \downarrow b' & & & \\ & C_0(A) & \xleftarrow{1-\lambda=0} & C_0 & \xleftarrow{1} & C_0 & \end{array}$$

$$b' = \sum_0^{n-1} (-1)^j (a_0, \cdots, a_j a_{j+1}, \cdots a_n)$$

$$\lambda = (-1)^{n+1} (a_n, a_0, \cdots)$$

horizontal is acycl. except for the first col.

$\rightsquigarrow$ $CC_*(A) \sim_{qis} (C_*(A)/(1-\lambda), b)$

The dual cplx will be $(\ker(1-\lambda), b) \subset C^*(A)'$

Prop cyclic cocycles $\equiv$ closed gr traces on $\Omega^*(A)$

Another interior prod: D deriv, ϕ: D-inv. cycl. cocyc

$$i_D \phi (a_0\, da_1 \cdots da_{n+1}) = \frac{1}{n+1} \sum^{n+1} (-1)^j \phi(a_0 da_1 \cdots D(a_j) \cdots)$$

Lem $i_D\phi$ is Hochschild-cohom to $\phi(D(a_{n+1})a_0, a_1, \cdots)$

Lem $i_D\, b\phi = b(\text{something})$, $i_D \phi \sim_{\text{Hoch}} \iota_D \phi$

Lem ϕ: D-inv $\rightsquigarrow$ $i_D\phi$ is a cyclic cochain.

Prop $\iota_D^2 \phi$ is trivial

suite : proof of Prop

since $\phi \in \ker(1-\lambda)$ in the first column,

$B_D \phi = 0$ and $\partial_D \phi = b_D \phi$: $\phi(D(a_{n+1})a_0, \cdots, a_n)$

$\rightsquigarrow \partial_D^2 \phi = \phi(D(a_{n+1})D(a_{n+2})a_0, \cdots)$

$\quad + \sum_{0 \le j \le k \le n-1} (-1)^{\cdot} \phi(D(a'_j), a'_{j+1}, \cdots D(a'_{k+1}), \cdots)$

Put $D \cup D = D(a_1) D(a_2)$

$\rightsquigarrow$ The first term in $\partial_D^2 \phi = b_{D \cup D} \phi$

The second part is $\sum \phi(a'_0, \cdots, D(a'_j), \cdots, D(a'_{k+1}), \cdots)$ by cyclicity.

the D-invariance impls this is eq. to $-\frac{1}{2} L_{D \circ D} \phi$

$\rightsquigarrow \partial_D^2 \phi = (\partial_{D \cup D} - \frac{1}{2} L_{D \circ D}) \phi$.

But $D \cup D = S(\frac{1}{2} D \circ D)$!

Cyclic cocycle on deformations

ϕ : $\mathbb{T}^2$-inv cyclic cocycle on $\mathbb{C}[\mathbb{T}^2_\theta]$

$\rightsquigarrow \phi^{\theta'}(a_0, \cdots, a_n) = \phi(a_0 \circ_{\theta'} da_1 \circ_{\theta'} \cdots \circ_{\theta'} da_n)$

is closed gr trace on $\Omega^*(\mathbb{C}[\mathbb{T}^2_{\theta+\theta'}])$

Thm $\phi \rightsquigarrow \phi^\theta + \theta \partial_{\partial_1} \partial_{\partial_2} \phi^\theta$ is the Monodromy of the Gauss-Manin connection.

Pf. ① $(1 + \theta \partial_{\partial_1} \partial_{\partial_2})$ is multiplicative ($\because \partial_{\partial_i}^2 = 0$)

② derivative involves

- $\partial_\theta \phi^\theta = \sum_{j \le k} \phi(a_0, \cdots, \partial_2 a_j, \cdots, \partial_1 a_{k+1}, \cdots)$

- $\partial_{\partial_1} \partial_{\partial_2} \phi = \phi(\partial_2(a_{n+1}) \partial_1(a_{n+2}) a_0, \cdots) \in \partial_{\partial}$

$\quad + \underbrace{B_{\partial_1} b_{\partial_2} \phi}_{\partial_\theta \phi^\theta}$

Generalizations

① ~~high~~ higher dim tori
(deform param = skew symm matrix)

② group alg $\mathbb{C}[\Gamma]$, $\omega_0 \in H^2(\Gamma; \mathbb{C})$
$\rightsquigarrow$ $e^{i\omega_0 \theta} \in H^2(\Gamma; \mathbb{C}^\times)$ $\theta \in \mathbb{C}$
cont family.