T. Crisp   Cyclic seminar

Cuntz - Quillen " Cyclic homology & nonsingularity " '95

- Motivation

  V : nonsmooth affine complex variety

  what is / how do we capture $H^*_{sing}(V;\mathbb{C})$?

Th'm (Grothendieck) If V happens to be smooth,

$$H^*_{sing}(V;\mathbb{C}) \simeq H^*(\underbrace{\Omega^*_K(V)}_{\text{Kähler diff cplx}})$$

Th'm (Deligne, Hartshorne) Generally, take embedding

$V \hookrightarrow S$ smooth. $H^*_{sing}(V;\mathbb{C}) \simeq H^* \Omega^*_K(S|_{\text{formal neighborhood}(V)})$

Idea "enrich" V $\gamma \rightsquigarrow$ 

- Dictionary

V: variety	A: algebra
H^*_{sing}	HP_*
$V \hookrightarrow S$	$A \twoheadleftarrow R$ quasi free $\hookleftarrow I$
Ω^*_K	X_*
formal neigh.	I-adic completion

Th'm (Cuntz - Quillen)

A unital alg / $\mathbb{C}$, R "quasi free" algebra

$A = R/I$ Then $HP_*(A) \simeq H_*(\varprojlim X_*(R/I^n))$

- Recalling HP, X, quasi-free algs

  A unital assoc alg over $\mathbb{C}$ (or k w/ chr 0)

  $\Omega^n = \Omega^n(A) := A \otimes (A/\mathbb{C}1)^{\otimes n}$ "nc-diff forms"

  $$a_0 \otimes a_1 \otimes \cdots \otimes a_n \longleftrightarrow a_0\, da_1 \cdots da_n$$

  - Ω is a dga
  - important operators:

    Hochschild boundary $b : \Omega^n \to \Omega^{n-1}$

    $b(\omega\, da) := (-1)^{|\omega|}[\omega, a]$

    Karoubi operator $\kappa : \Omega^n \to \Omega^n$, $1 - [b, d]$

    $(\kappa(\omega\, da) = (-1)^{|a|} da\, \omega$

    Connes / Rinehart coboundary $B : \Omega^n \to \Omega^{n+1}$

    $B = \sum_{i=0}^{n} \kappa^i d$.

Lemma $b^2 = B^2 = (b+B)^2 = 0$

$\rightsquigarrow$ Ω, $\hat{\Omega} = \prod \Omega^n$ are $\mathbb{Z}/2$-graded cplexes

$$\Omega^{ev} \underset{b+B}{\rightleftarrows} \Omega^{odd}$$

Dfn $HP_*(A) = H(\hat{\Omega}, b+B)$, $HH_*(A) = H_*(\Omega, b)$

periodic cyclic homology

Dfn $F_n := b\Omega^{n+1} \oplus \Omega^{n+1} \oplus \Omega^{n+2} \oplus \cdots$

$\rightsquigarrow$ $\hat{\Omega} = \varprojlim \Omega / F_n$

The X-complex : $X_*(A) = \Omega / F_1$

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Proposition Suppose $HH_n A = 0$ for $n \geq 2$

Then $HP_*(A) = H_*(X(A))$

Pf: The $b+B$ - cplx F_{n-1}/F_n is $b\Omega^n \underset{b}{\overset{0}{\rightleftarrows}} \Omega^n / b\Omega^{n+1}$

$$\begin{array}{lll|lll} F_{n-1} & b\Omega^n & \Omega^n & F_n & \Omega^{n+1} & b\Omega^{n+1} \\ & \Omega^{n+1} & \Omega^{n+2} & & \vdots & \Omega^{n+2} \\ & \vdots & \vdots & & \vdots & \vdots \end{array}$$

$\rightsquigarrow$ $H_0(F_{n-1}/F_n) = 0$, $H_1(F_{n-1}/F_n) = HH_n(A)$

From $F_{n-1}/F_n \to \Omega/F_n \to \Omega/F_{n-1}$

$\rightsquigarrow$ $\Omega/F_3 \to \Omega/F_2 \to \Omega/F_1$ is "const in homology" $\square$

In other words $X(A): A \underset{b}{\overset{d}{\rightleftarrows}} \Omega^1/[A, \Omega^1]$

Proposition Let A be an algebra : TFAE

① $\begin{array}{ccc} & & B \\ & \overset{\exists}{\nearrow} & \downarrow \\ A & \to & B/I \end{array}$ I nilpotent

② $H^2(A, M) = 0$ for $\forall$ A-bimod M

Dfn A is said to be quasi-free if it sats ①/②

Remark ② $\Rightarrow$ $HH_{\geq 2} A = 0$

- If V is smooth, $\mathcal{O}(V)$ satisfies ① in the category of comm algs, but $\mathcal{O}(V)$ is q-free only when $\dim V \leq 1$

Examples free algs $\mathbb{C}$, $\mathbb{C}\langle x_1, \cdots, x_n \rangle$

$\mathbb{C}$ ⇝ $\mathbb{C} \rightleftarrows 0$, $\mathbb{C}\langle x_1, \cdots, x_n \rangle$ ⇝ $\mathbb{C} \rightleftarrows 0$ $\oplus$ (contractible)

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suite $A = \mathbb{C}[z, z^{-1}]$ is quasi-free

$\because$ one can lift invertible elements in nilpotent extension.

$\rightsquigarrow$ $X(A) = A \rightleftarrows \Omega^1_K(A)$

$\rightsquigarrow$ $H_0 \cong H_1 \simeq \mathbb{C}$

$\mathbb{C}[\mathbb{F}_n]$ is q-free.

Exercise: compute $HP_* \mathbb{C}[\mathbb{F}_n]$ from the X-cplx

(Answer $H^*(B\mathbb{F}_n, \mathbb{C})$)

- HP_* via quasi-free extensions

universal extension:

<u>Dfn</u> $R(A) = \Omega^{even}(A)$ w/ the Fedsov prod

$\omega \circ \eta = \omega\eta - d\omega\, d\eta$

$\rightsquigarrow$ extension $IA = \bigoplus_{n \geq 1} \Omega^{2n} \to RA \to A$

Prop
- $RA \simeq T(A/\mathbb{C})$ free
- $\Omega^1(RA)/[RA, \Omega^1(RA)] \xrightarrow{\sim} \Omega^{odd}(A)$
- $X(RA) : \Omega^{even}(A) \underset{\beta}{\overset{\delta}{\rightleftarrows}} \Omega^{odd}(A)$

Recall the filtration $F_n \subset \Omega$

Lem. $F_n \subset X(RA)$ is also a subcplx

$\varprojlim X(RA)/F_n \simeq \varprojlim X(RA/IA^n)$

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We need to compare $(\Omega, \delta \oplus \beta)$ and $(\Omega, b+B)$

Recall the Karoubi operator $\mathcal{K} = 1 - [b, d]$

Lem $(\mathcal{K}^n - 1)(\mathcal{K}^{n+1} - 1) = 0$ on Ω^n

Cor $\Omega = \underset{\ker (\mathcal{K}-1)^2}{p\Omega} \oplus p^\perp \Omega$ $(\mathcal{K}-1)$ is invertible.

On $p^\perp\Omega$, $[b, d]$ is invertible.

$\rightsquigarrow [b+B, [b,d]^{-1} d] = [b,d]^{-1}[b+B, d] = 1$

commute

$\rightsquigarrow (p^\perp\Omega, b+B)$ is contractible.

similarly for $(p^\perp \Omega, \delta + \beta)$

Thm $(p\Omega, b+B) \simeq (p\Omega, \delta \oplus \beta)$

by $n! \times$ on $\Omega^n[A]$.

as filtered cplxes

Finishing argument,

$HP_* A = H_* \varprojlim \Omega^{b+B}/F_n = H_* \varprojlim \Omega^{\delta+\beta}/F_n = H_* \varprojlim X(RA/IA^n)$

What about general quasi free extensions ?

Thm Let $A = R_{q\text{-free}}/I$, $B = S_{q\text{-free}}/J$

any $\varphi : A \to B$ lifts to

$\varphi_* \quad \varprojlim X(R/I^n) \longrightarrow \varprojlim X(S/J^n)$

which is unique up to chain homotopy

Cor Cuntz-Quillen

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Thm (Goodwillie) $A \to B$ nilpotent extension

$$HP_* A \xrightarrow{\sim} HP_* B$$

Pf.

$$\begin{array}{ccccc} I & \to & R \ q\text{-free} & \to & A \\ \cap & & & & \downarrow \\ J & \to & R & \longrightarrow & B \end{array} \quad \rightsquigarrow \quad X(R/I^n) \simeq X(R/J^n)$$