Markus Cyclic seminar

§ Motivation

Symmetry in pos char p

"Frobenius $F(a) = a^p$"

$\rightsquigarrow$ • $Gal(\bar{\mathbb{F}}_p) \simeq \hat{\mathbb{Z}}$ gen'ed by F

• center of the category $\mathbb{F}_p$-alg $\simeq \mathbb{N}$ gen'ed by F

center of category $\mathcal{C}$

$$Z(\mathcal{C}) = Nat(Id_{\mathcal{C}}, Id_{\mathcal{C}}) = \{ (\underset{X \to X}{\phi_X} : X \in \mathcal{C}) \quad \underset{\phi_y f = f \phi_x}{\forall f\, x \to y} \}$$

"Deformation" in characteristic p

0. generalize to simplicial $\mathbb{F}_p$-algs
1. what do we want to ask?
2. answer?

Recall: From simplicial cat $\mathcal{C}$,

$\rightsquigarrow$ $Z(\mathcal{C})$: simplicial comm monoid (too rigid)

$\dashrightarrow$ $\zeta(\mathcal{C}) = Tot(\Pi^{\bullet}\mathcal{C})$

where $\Pi^n \mathcal{C} = \prod_{X_0, \dots, X_n} Map(\mathcal{C}(X_1, X_0) \times \dots \times \mathcal{C}(X_n, X_{n-1}), \mathcal{C}(X_n, X_0))$

elems : fam (ϕ^n)

e.g. $\phi^0 = (\phi^0_X : X \to X, X \in \mathcal{C})$

$\phi^1_{x,y}$: $\forall f\, x \to y$, homotopy

$\phi_y f \simeq f \phi_x, \dots$

Th'm $\zeta(sAlg(\mathbb{F}_p)) = \mathbb{N}$ gen'ed by Frob

§ How to calculate

Generalization to $F, G : \mathcal{C} \to \mathcal{D}$

$\Pi^n(F,G) = \prod Map(\mathcal{C}(X_1, X_0) \times \cdots \times \mathcal{C}(X_n, X_{n-1}), \mathcal{D}(FX_n, GX_0))$

$\mathcal{N}(F,G) = \mathrm{Tot}(\Pi^*(F,G))$

For "h-fully faithful" functors $F : \mathcal{C} \to \mathcal{D}$

($\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is h-equiv)

one has $\mathcal{N}(FG, FH) \simeq \mathcal{N}(G,H)$

in particular, $\zeta(\mathcal{C}) \simeq \mathcal{N}(F.F)$

· Left Kan extension :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{Q} & \mathcal{E} \\ {\scriptstyle F}\downarrow & & \\ \mathcal{D} & & \end{array}$$

We say $\mathcal{L}Kan_F Q$ exists $\mathcal{D} \to \mathcal{E}$

if $\mathcal{N}(\mathcal{L}Kan_F Q, R) \simeq \mathcal{N}(Q, RF)$

for any $R : \mathcal{D} \to \mathcal{E}$

$F : \mathcal{C} \to \mathcal{D}$ is <u>dense</u> iff $id_{\mathcal{D}} \simeq \mathcal{L}Kan_F F$

Prop. $F : \mathcal{C} \to \mathcal{D}$ dense, $G : \mathcal{D} \to \mathcal{D}$

$\Rightarrow \mathcal{N}(Id_{\mathcal{D}}, G) \simeq \mathcal{N}(F, GF)$

$\because$ adjointness

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suite: Cor F: dense $\to$ $\zeta(\mathcal{D}) \simeq \eta(F, F)$

F: h-f.f, dense $\Rightarrow$ $\zeta(\mathcal{C}) \simeq \zeta(\mathcal{D})$

§ Presheaves

$\mathcal{C}$ simplicial cat

$sPre(\mathcal{C}) = (\mathcal{C}^{op} \to (\text{spaces}))$

Yoneda emb $\mathcal{C} \to sPre(\mathcal{C})$, $X \to \mathcal{C}(-, X) = Y_{\mathcal{C}}(X)$

$\leadsto$ $sPre(Y_{\mathcal{C}}(X), P) = P(X)$

$\Rightarrow$ $Y_{\mathcal{C}}$ is h-fully faithful

Prop. F is dense iff $\mathcal{D} \xrightarrow{Y_{\mathcal{D}}} sPre(\mathcal{D}) \xrightarrow{F^*} sPre(\mathcal{C})$

is h-f.f.

Cor $Y_{\mathcal{C}}$ is dense

Cor-Thm $\zeta(sPre(\mathcal{C})) \simeq \zeta(\mathcal{C})$

§ Localization

left Bousfield localization

adjunction $I : \mathcal{L} \rightleftarrows \mathcal{J} : J$ s.t. J is h-f.f.

$\Rightarrow$ $\begin{cases} IJ \,\hat{=}\, Id_{\mathcal{J}} \\ JI =: L \quad \text{" localization functor "} \end{cases}$

Prop I is dense.

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Th'm $\zeta(\mathcal{J}) \simeq n(\mathrm{Id}_{\mathcal{L}}, L)$

$\because$ adjointness + $\mathcal{L}(Lx, Ly) = \mathcal{L}(x, Ly)$

Th'm $I : s\mathrm{Pre}(\mathcal{C}) \rightleftarrows \mathcal{J} : J$

if representatives are local, $\zeta(\mathcal{J}) \simeq \zeta(s\mathrm{Pre}(\mathcal{C}))$

$\because$ $L Y_c \simeq Y_c$ by hypo

§ Algebraic theories

alg theory : cat $\mathcal{C}$ w/ coproducts

objs enum by $\mathbb{N}$: $a_0, a_1, a_2, \dots$

$a_{m+n} \simeq a_m + a_n$

" a_n is the free Θ-alg on n-gens "

$s\mathrm{Alg}(\Theta) := \{ P \in s\mathrm{Pre}(\Theta) : P(x+y) \simeq P(x) \times P(y) \}$

Th'm (Bodrioch) $\exists$ " $h\mathrm{alg}(\Theta)$ " s.t.

$$\begin{tikzcd} s\mathrm{Alg}(\Theta) \arrow[r, "\simeq"] \arrow[dr, hook] & h\mathrm{Alg}(\Theta) \arrow[d, shift left, "J"] \\ & s\mathrm{Pre}(\Theta) \arrow[u, shift left, "I"] \end{tikzcd}$$

Th'm $\zeta(s\mathrm{Alg}(\Theta)) \simeq \zeta(h\mathrm{Alg}(\Theta)) \simeq \zeta(s\mathrm{Pre}(\Theta))$

$\simeq \zeta(\Theta) \simeq \zeta(\mathrm{Alg}(\Theta)) = Z(\mathrm{Alg}(\Theta))$