R. Nest 2013.03.15

A unital k-algebra $\quad \mathrm{chr}(k) = 0$

$C_n(A) = A \otimes (A/k)^{\otimes n}$

$b : C_n(A) \to C_{n-1}(A)$, $\quad B : C_n \to C_{n+1}$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

is the Hochschild differential $\quad H_*(C.(A), b) = HH.(A) = \mathrm{Tor}_{A^e}(A, A)$

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$

is the Rinehard differential

Exercise: show $B^2 = 0$, $(b+B)^2 = 0$

$$\begin{tikzcd}
 & & \deg 1 \quad C_0(A) \\
b\uparrow \xrightarrow{B} & \deg 0 \quad C_0(A) \arrow[r] & C_1(A) \arrow[u] \\
\deg(-1) \quad C_0(A) \arrow[r] & C_1(A) \arrow[r] \arrow[u] & C_2(A) \arrow[u]
\end{tikzcd}$$

The negative cyclic complex $\quad CC_n^-(A) = \prod_{\substack{i \geq -n \\ i \equiv n\,(2)}} C_i(A)$

The cyclic cplx $\quad CC_n(A) = \bigoplus_{\substack{i \leq -n \\ i \equiv n\,(2)}} C_i(A) \quad (-\mathbb{N})$-graded

The periodic cyclic complex $\quad CC_n^{per}(A) = \prod_{i \equiv n\,(2)} C_i(A)$

short exact seq. $\quad 0 \to CC_*^-(A) \to CC_*^{per}(A) \to CC_*(A)[2] \to 0$

and, $\quad 0 \leftarrow CC.(A)[2] \leftarrow CC.(A) \leftarrow C.(A) \leftarrow 0$

Connes – Gysin exact seq.

Motivation if $A = C^\infty M$ for cpt M,

we have $A \hat{\otimes} A = C^\infty(M \times M)$ etc.

do the "same" constructions w/ $\hat{\otimes}$ $A^{\hat{\otimes} n} = C^\infty(M^n)$

also, $(A \otimes \bar{A}/k)^{\otimes n}, b)$ and $(A \otimes A^{\otimes n}, b)$ are homotopic.

Hochschild - Kostant - Rosenberg
Connes Soibelmann, Voronov $\Big\} \Rightarrow HH_*(A) = \Omega^\bullet(M)$

$A^e = A \otimes A^{op}$ A is an A-bimodule

Hochschild complex can be thought as a free A^e-res $(A^e \otimes A^{\otimes n}, b)$ tensored w/ A

another resolution $P_n = C^\infty(M) \hat{\otimes} \Omega^n(M)$

Assume $\chi(M) = 0$ (otherwise use $M \times S^1$ instead)

we can take $V \in \mathrm{Vect}(M \times M)$ nonvan everywhere off $\Delta(M)$

transverse to $\Delta(M)$ of the form "$x_1 - x_2$"

contraction by V gives diff $P_n \xrightarrow{\iota_V} P_{n-1}$

it is contraction $\iota_V \iota_V^* + \iota_V^* \iota_V = \|V\|^2$

$(P_\cdot, \iota_V)$ computes $HH_\cdot(A)$ (?)

exactness of the $(P_\cdot, \iota_V)$ comes from the Taylor expansion $\Rightarrow f(x_1, x_2)|_{\Delta M} = 0$

" $(x_1 - x_2) \mid f$

(not possible in continuous category!)

$(P_\cdot, \iota_V) \otimes_{A^e} A$ is $(\Omega^\bullet(M), \underbrace{\iota_V|_{\Delta(M)}}_{"0})$

The spectral sequence

vertical cohom for:

$$\begin{array}{ccc} & & C_0 \\ & & \uparrow b \\ & C_0 \dashrightarrow & C_1 \\ & b\uparrow & \uparrow b \\ \dashrightarrow & C_1 \dashrightarrow & C_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{cc} & \Omega^0 \\ \Omega^0 & \Omega^1 \\ \Omega^1 & \Omega^2 \end{array}$$

this can be expressed as (Loday - Quillen)

$$C_n(A) \ni f_0 \otimes \cdots \otimes f_n \longmapsto \frac{1}{n!} f_0 df_1 \cdots df_n \in \Omega^n M$$

A is comm: this is a cplx map from b to 0

The B becomes the de Rham differential

Any cycle of $(\Omega^\cdot, d)$ comes from a <u>cycle</u> of $(C_\cdot A, b)$ $\rightsquigarrow$ no higher diffs

$$\rightsquigarrow \quad HC_n(C^\infty M) = \bigoplus_{\substack{k \equiv n\,(2) \\ k < n}} H^k_{DR}(M) \oplus \Omega^n / d\,\Omega^{n-1}$$

$$HC^{per}_n(C^\infty M) = \bigoplus_{k \equiv n\,(2)} H^k_{DR}(M)$$

$$(CC^{per}_*(A),\ b + B)$$
$$\Big\downarrow \ E^1\text{-term}$$
$$(\Omega^\cdot M,\ d_{DR}) \qquad : (HH_\cdot(A),\ B)$$
$$\Big\} \text{ collapse}$$

this might be too small for noncommutative algs

If A is not regular (contains

$$HC^{per}_*(A) \longrightarrow \bigoplus H^*_{crys}(M)$$

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Noncommutative torus : unitaries u, v s.t. $uv = e^{i\theta} vu$

$\{ \sum a_{m,n} u^m v^n \}$ rapid decay $=: C^\infty(T_\theta)$

$$\rightsquigarrow \quad HH_*(\mathbb{T}^2_\theta) = \begin{cases} \Omega^\bullet(\mathbb{T}^2/\sim) & \frac{1}{2\pi}\theta \in \mathbb{Q} \\ H^*_{DR}(\mathbb{T}^2) & \frac{1}{2\pi}\theta \notin \mathbb{Q}, \text{ generic.} \\ H^*_{DR}(\mathbb{T}^2) \oplus \boxed{\sim\!\sim} & \frac{1}{2\pi}\theta \notin \mathbb{Q}, \text{ singular} \end{cases}$$

behavior of $\dfrac{1}{1-\theta^n}$ matters !

anyhow, $\quad HC^{per}_*(\mathbb{T}^2_\theta) \simeq H^\bullet_{DR}(\mathbb{T}^2)$

Getzler-Jones notation :

throw in an extra variable $\deg u = -2$.

$$\begin{array}{llll} & \deg(-2) & & u^2 C_0 \\ & & u C_0 & \xrightarrow{uB} u^2 C_1 \\ \deg 0 & & \uparrow b & \\ & C_0 & u C_1 & u^2 C_2 \end{array} \qquad \rightsquigarrow \quad (C_*(A)[u^{-1}, u]], \; b + uB)$$

$$\wr$$

$$(\Omega^*(M)[u^{-1}, u]], \; ud)$$