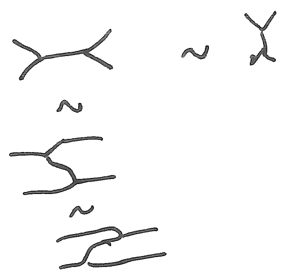
Constructing operations on HH_* of comm. Frobenius algebras

Def: Frobalg: • Algebra $(A, m, \mathbb{1}_A)$

• Coalg (A, Δ, η)

s.t. $\Delta \circ m = (m \otimes id) \circ (id \otimes \Delta)$

$= (id \otimes m) \circ (\Delta \otimes id)$



symm: $\eta \circ m \circ \tau = \eta \circ m$

$\Uparrow$

comm: $m \circ \tau = m$

SymmFA $\simeq H_0(\mathcal{O}\ )$ $\qquad$ $\mathcal{O}(n,m)$ fat graphs, computes hom. of the mod. space of Riemann surf. from n to m interv.

one free gen. for each top. type of open cob.

CommFA extra relation

Main Question

$$CC_*(A)^{\otimes n_1} \otimes A^{\otimes m_1} \longrightarrow CC_*(A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

" $\approx Nat_{\mathcal{E}}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$ " $\qquad$ $\mathcal{E}$ class of algebras

Motivation: Field, M formal, M 1-conn.

$HH_*(H^*(M), H^*(M)) \cong H^*(LM)$

$\rightsquigarrow H^*(LM)^{\otimes n_1} \otimes H^*(M)^{\otimes m_1} \to H^*(LM)^{\otimes n_2} \otimes H^*(M)^{\otimes m_2}$

" string topology "

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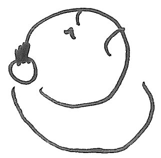
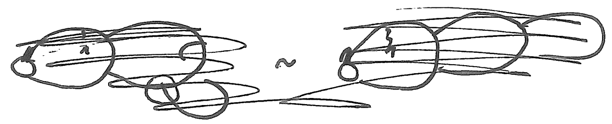
i) SymFr $\qquad$ Thm [Wahl]: $Nat_{SymFr}\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right) \simeq SD\left(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]\right)$

Sullivan diagrams





ii) Comm. algebras

a) Loday's shuffle op. in $\mathrm{Nat}_{\mathrm{Com}}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$

sh_k is 0 on degree $\leq k$.

(example $(0, id, id, \dots)$)

b) Shuffle product in $\mathrm{Nat}_{\mathrm{Com}}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$

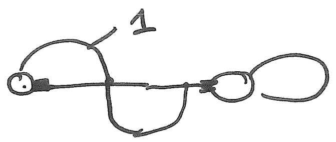
$\left(a_0 b_0, \frac{\text{shuffles}}{}\right)$

<u>Goal</u>: Find a cplx that encodes both Phenomena!

Graphs

$\binom{p}{n}$ - black and white Graph (BW-graph)

- all vertices labeled black or white
- p white
- black at least 3-valent
- ordering at vertices
- white vertex has a start edge
- n-labeled leaves

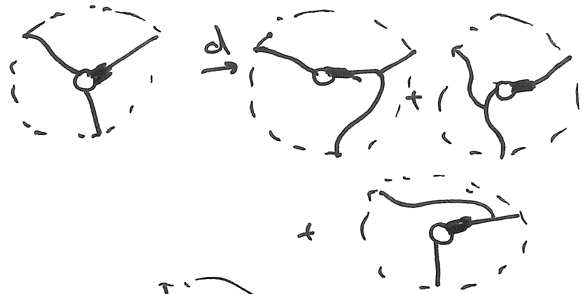


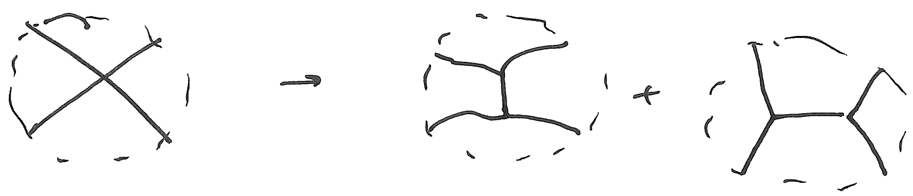
$\binom{2}{1}$-BW graph

degree 3

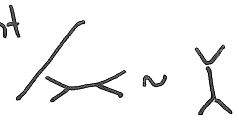
degree at white vertex = edges − 1
black " − 3

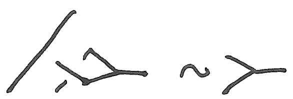
Differential: blow up of vertices
(black still 3-valent)





$\binom{p}{n}$ - Sullivan diagr.

$\binom{p}{n}$ - BW / < >3-valent black vert. > = 3-valent / 
+ their bdry

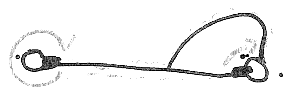
$\binom{p}{n}$ - Comm. SD = SD /  forget ordering at black vert.

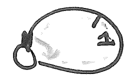
Looped Diagrams

loop in a Comm. SoD :

- coll. of segm. of bdry cycles of white vertices s.t. gives loop when walking through arcs

from leaf : From arc. comp. with that leaf





$\binom{p}{(m,n)}$ - looped diagram $(\Gamma, \gamma_1, \dots, \gamma_n)$

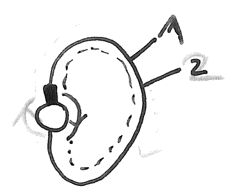
Γ : $\binom{p}{m+n}$ - comm SoD

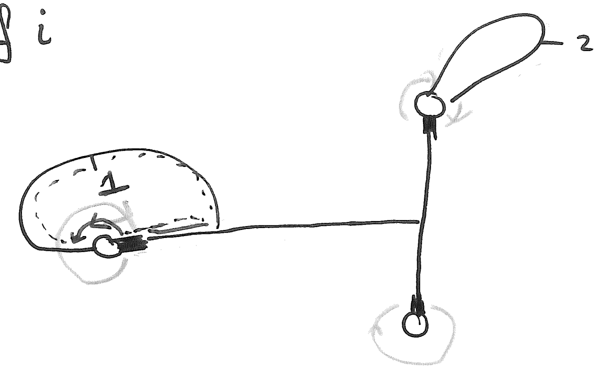
γ_i : loop starting from leaf i

Examples: (Paper)

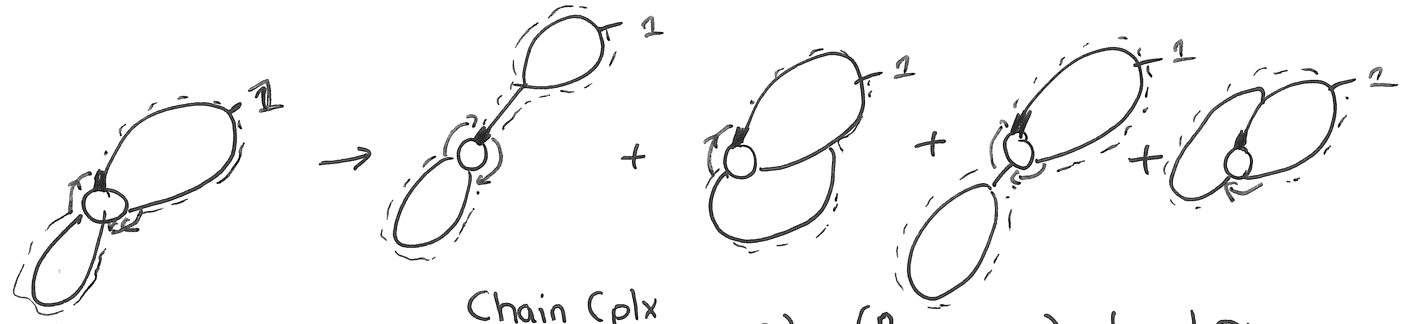
~~Diff : Paper~~

~~Comp : Paper~~





Differential: Blow up

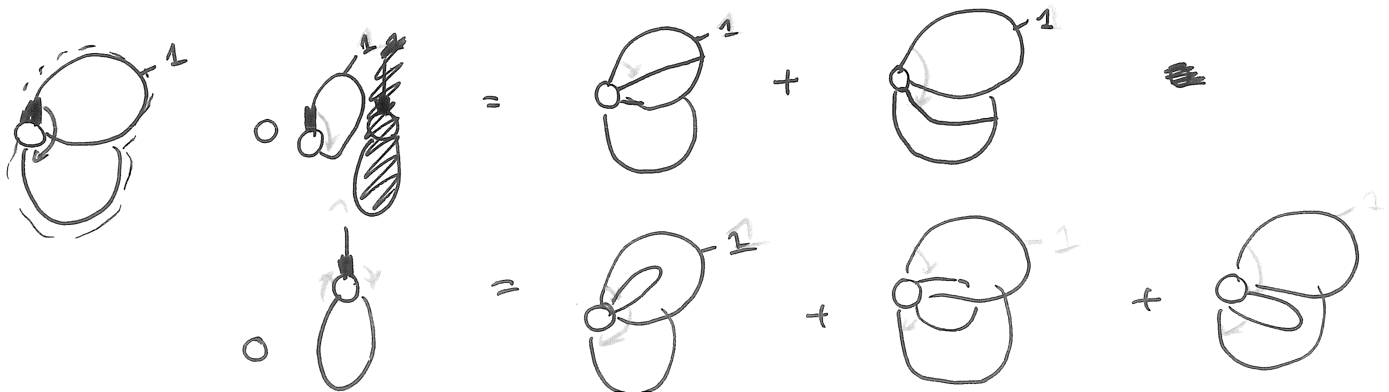


Chain Cplx

$LD\left(\left[{}^{n_1}_{m_1}\right], \left[{}^{n_2}_{m_2}\right]\right) = \binom{n_2}{(m_1+m_2,\ n_1)}$ - Looped Diagrams

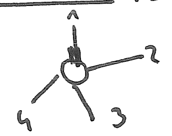
Non-const :

Composition : Put at all pos. places along loop



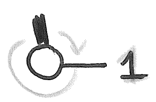
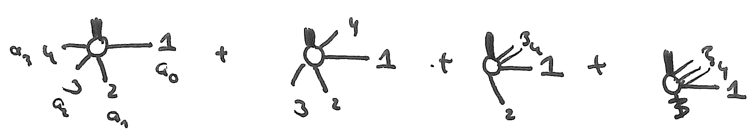
Prop: $L\mathcal{D}\left(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right)$ is a dg-category.

Construction of operations

$l_j =$  $\in L\mathcal{D}\left(\begin{bmatrix} 0 \\ j \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

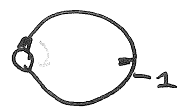
$$G^\circ \left(l_{j_1} \amalg \ldots \amalg l_{j_{n_1}} \amalg id_{m_1} \right) \in L\mathcal{D}\left(\begin{bmatrix} 0 \\ j_1 + \ldots + j_{n_1} + m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right)$$

Operation on degree $(j_1, \ldots, j_{n_1})$!

BV (alr. in Sullivan)  $\circ\, l_4$ 

Shuffle op. 

"non-empty"



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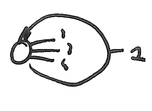
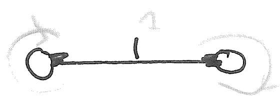
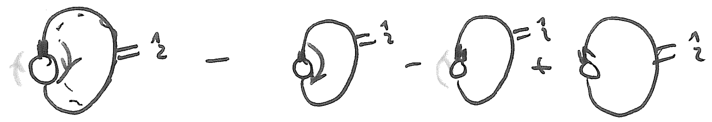
 $=$ "$a_0 \otimes \ldots \otimes a_j \mapsto \sum a_0' \otimes a_1 \otimes \ldots \otimes a_j \otimes a_0''$"

Thm: Functor of dg-categories

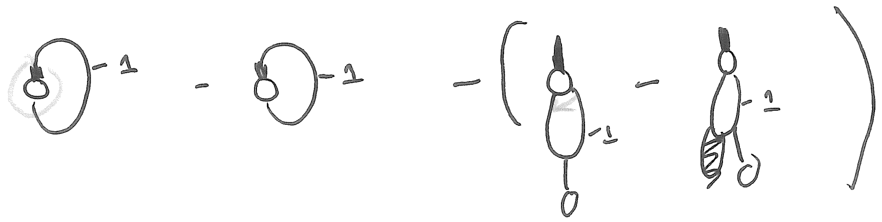
$$L\mathcal{D} \to Nat_{CFr}$$

Easier to find relations:

Copr. 

$m =$ 

$$m \circ Copr = d(D)$$

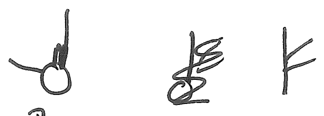
$D =$ 

Results: Allow products of diagrams over types

$\prod_{\text{types}} \mathcal{LD}_s \to \mathrm{Nat}$

Commutative: $\mathrm{Nat}_{Com} \hookrightarrow \mathrm{Nat}_{\cancel{Opr}}$

$\mathcal{LD}_{Com}\left(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix}\right)$



trees on n_2 vert

+ m_2 outg leaves

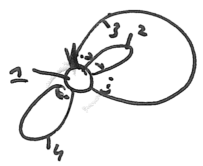
Thm: $\widetilde{\mathcal{LD}}_{Com} \longrightarrow \mathcal{LD}_{Com} \simeq \mathrm{Nat}_{Com}$

not cat but triv. diff.

Cacti: Note: partly const. diagr. split subcplx

$\mathcal{LD}^{>0}$ quotient

$\mathcal{LD}^{>0}_{Cact}$:
- white vert not conn
- Γ emb. into plane
- every bdry segm. in ex 1 loop
- all loops irred



Cacti Operad

Normalized all lopes length 1

Thm: $H_*(\mathcal{LD}^{>0}_{Cact}(\qquad)) \cong s^{-1}H_*(Cact)$