§ Motivations 1. the NC-torus $\mathbb{C}[T^2_\theta] = \langle u, v : uv = e^{i\theta} vu \rangle$

$\rightsquigarrow$ $HP_*(\mathbb{C}[T^2_\theta]) = \mathbb{C}^2$ $\quad * = 0, 1$

Q. Is there some natural way to identify $HP_* \mathbb{C}[T^2_\theta] \simeq HP_* \mathbb{C}[T^2_{\theta'}]$

It should define a 'connection' over the vector bundle

$(HP_* \mathbb{C}[T^2_\theta])_\theta \to$ "parameter space for θ"

s.t. the parallel transport gives the nat. isoms.

How about the more general 'smooth family' of algs ?

2. Classical Cartan homotopy formula

M manifold, $X \in \Gamma TM$ vector field over M

① Lie derivative $L_X \curvearrowright \Omega^* M$

$$L_X (f^0 df^1 \cdots df^n) = X(f^0) df^1 \cdots df^n + \sum_{j=1}^{n} f^0 df^1 \cdots dX(f^j) \cdots df^n$$

② interior product $\iota_X \curvearrowright \Omega^* M$ (contraction)

$$\iota_X (f^0 df^1 \cdots df^n) = \sum_{j=1}^{n} (-1)^j f^0 X(f^j) df^1 \cdots \hat{j} \cdots df^n$$

③ Cartan homotopy formula : $L_X = d\iota_X + \iota_X d$.

$\Rightarrow$ If X generates a $U(1)$ action (discrete spectrum),

$H^* M = H^*(\ker L_X, d)$ (X-inv. diff forms)

3. Deformation quantization

Polyvector field $X \in \Gamma \wedge^\bullet TM$ defines $L_X, \iota_X \curvearrowright \Omega^* M$.

satisfying the CHF.

Poisson bivector field $\pi \in \Gamma \wedge^2 TM \rightsquigarrow$ new diff $d + L_\pi$.

$A_\hbar$: deform. quant. alg. of $C^\infty(M)[[\hbar]]$

$$\frac{1}{\hbar}[f^0, f^1]_{A_\hbar} \longrightarrow (\pi, df^0 \wedge df^1)$$

$\rightsquigarrow$ $HP_*(A_\hbar) \xrightarrow{\simeq} H^*(\Omega^*(M)[[\hbar]], d + \hbar L_\pi)$

$\updownarrow e^{\hbar \iota_\pi}$ ($\because$ CHF).

$HP_*(C^\infty M[[\hbar]]) \simeq H^*(\Omega^*(M)[[\hbar]], d)$

§ Hochschild cochains (w/ coeff. A)

$C^n := C^n(A, A) = L(A^{\otimes n}, A)$ "NC-polyvector fields"

$\delta : C^n \to C^{n+1}$, $\delta(D)(a_1, \dots, a_{n+1}) = a_1 D(a_2, \dots, a_{n+1}) - D(a_1 a_2, a_3, \dots, a_{n+1})$
$+ \dots + (-1)^n D(a_1, \dots, a_n a_{n+1}) + (-1)^{n+1} D(a_1, \dots, a_n) a_{n+1}$

pre-Lie product of $D \in C^m$ and $D' \in C^n$; $D \circ D' \in C^{m+n-1}$

$$D \circ D'(a_1, \dots, a_{m+n-1}) = \sum (-1)^{j(n-1)} D(a_1, \dots, D'(a_{j+1}, \dots, a_{j+n}), \dots,)$$

Gerstenhaber bracket $[D, D']_G = D \circ D' - (-1)^{(m-1)(n-1)} D' \circ D$.

Conceptually: $T^*A = \bigoplus_{n=0}^{\infty} A^{\otimes n}$ graded coalgebra by

$$\Delta(a_1, \dots, a_n) = \sum_{j=0}^{n} (a_1, \dots, a_j) \otimes (a_{j+1}, \dots, n)$$

$D \in C^n \rightsquigarrow$ degree $(n-1)$ coderivation on T^*A :

$$\tilde{D}(a_1, \dots, a_m) = \begin{cases} 0 & m < n \\ \sum_{j=0}^{m-n} (-1)^{j(n-1)} (a_1, \dots, D(a_{j+1}, \dots, a_{j+n}), \dots) \end{cases}$$

$$\Delta \tilde{D} = (\tilde{D} \otimes 1 + 1 \otimes \tilde{D}) \Delta .$$

$$\rightsquigarrow \widetilde{[D, D']_G} = [\tilde{D}, \tilde{D}']_{gr} .$$

Product structure $m \in C^2 \rightsquigarrow$ associativity $\equiv m \circ m = 0$.

$$\delta(D) = [m, D]_G .$$

$C^{*-1} = C^{*-1}(A, A)$ is a dg Lie alg ($[\sim, \sim]_G$, δ, $|D| = n-1$ for $D \in C^n$)

§ Actions on the Hochschild chains $C_k(A) = A^{\otimes k+1}$

① Lie derivative $D \in C^n$ $L_D : C_k \to C_{k-n+1}$

$$L_D(a_1, \dots, a_k) = \sum (-1)^{|D|(j+1)} (a_1, \dots, D(a_{j+1}, \dots, a_{j+n}), \dots)$$
$$+ \sum (-1)^{k(k-j)} (D(a_{j+1}, \dots, a_0, \dots), \dots, a_j)$$

($\uparrow$ contains a_0.)

! L_D does not depend on $m \in C^2$.

normalized cochains : $\bar{C}^k = \{ D \in C^k : D(a_1, \dots, 1, \dots, a_k) = 0 \}$

$D \in \bar{C}^k \rightsquigarrow L_D \curvearrowright \bar{C}_k(A) : A \otimes (A/\mathbb{C})^{\otimes k}$

• $(a_0, \dots, a_k) =$ "$a_0 da_1 \cdots da_k$", D and $\underset{\deg 1}{d}$ gr. commute

② Interior product of $D \in C^n$

"Cap product" : $b_D(a_0, \dots, a_k) = (-1)^{kn} (D(a_{k-n+1}, \dots, a_k) a_0, a_1, \dots, a_{k-n})$

"the other part" : $B_D(a_0, \dots, a_k) = \sum (-1)^{k(j+1)+(n-1)(k-j)} (1, a_{j+1}, \dots, D(a_{k+1}, \dots), \dots a_0 \dots)$

a_0 comes to the right of D.

$\iota_D := b_D - B_D \qquad (D \in \bar{C}^k \rightsquigarrow \iota_D \curvearrowright \bar{C}_*)$

Thm (Getzler's CHF) : $[b - B, \iota_D] = L_D - \iota_{\delta(D)}$.

Proof : compute !

§ Deformation

A : vec sp., $(m_t)_{t \in [0,1]}$ smooth family of assoc. alg. structures

$\gamma_t := m_t - m_0$.

the associativity cond. $m_t \circ m_t = 0$

$$\rightsquigarrow \underbrace{m_0 \circ \gamma_t + \gamma_t \circ m_0}_{[m_0, \gamma_t]_G = \delta(\gamma_t)} + \underbrace{\gamma_t \circ \gamma_t}_{\frac{1}{2}[\gamma_t, \gamma_t]_G} = 0$$

$$\rightsquigarrow \delta_{m_0}(\gamma_t) + \frac{1}{2}[\gamma_t, \gamma_t]_G = 0 \quad \text{(Maurer-Cartan eq.)}$$

derivation at $t = 0 \rightsquigarrow \delta_{m_0}(\dot{\gamma}_{t=0}) = 0$.

$\dot{\gamma}_0$ is a cocycle for (A, m_0)

$\mathcal{A} = C^{\infty}([0,1], A)$, $f \cdot g(t) = m_t(f(t), g(t))$.

∂_t is a cochain in $\bar{C}^1(\mathcal{A})$.

The Cartan homotopy formula gives:

$[b-B, \iota_{\partial_t}] = L_{\partial_t} - \iota_{\delta(\partial_t)}$ on $\mathcal{A}$.

the right hand side: $L_{\partial_t} : (a_0, \cdots, a_n) \mapsto \Sigma_j (a_0, \cdots, \dot{a}_j, \cdots, a_n)$

natural extension of ∂_t.

$\delta(\partial_t) = [m_{\mathcal{A}}, \partial_t] = -\dot{\gamma}_t$

$\Rightarrow$ $\partial_t + \iota_{\dot{\gamma}}$ acts as null-homotopic on $\bar{C}_*(\mathcal{A})$.

$\Rightarrow$ the img of $HP_*(\mathcal{A}) \to C^\infty([0,1], HP_*(A; m_t))$

satisfies $\partial_t c + \iota_{\dot{\gamma}} c = 0$.

$\iota_{\dot{\gamma}}$ is the connection form (Gauss-Manin conn.

§ Generalizations

① A_∞-algebra: $m_* = \Sigma m_k : m_k \in C^k(A, A)$, $m_* \circ m_* = 0$.

and, m_* reps a coderiv of deg. -1 on T^*A.

b_D becomes $\sum_{\substack{j \le p \\ 2 \le k}} (-1)^\epsilon (m_k(a_{j+1}, \cdots, D(a_{p+1}, \cdots), \cdots, a_0, \cdots), \cdots, a_j)$

$(C^*(A,A), \delta = [m_*, -])$ dg Lie alg.

$(\bar{C}_*(A), b = L_{m_*}, B)$ (normalized) (b,B)-complex.

L_D (the same formula), B_D (the same), $\iota_D = b_D - B_D$

$\rightsquigarrow$ CHF: $[b-B, \iota_D] = L_D - \iota_{\delta(D)}$.

$\rightsquigarrow$ Gauss-Manin conn.

② we want: up to homotopy,

$[L_D, L_{D'}] = L_{[D,D']}$, $\iota_{D \cup D'} = \iota_D \, \iota_{D'}$,

$L_{D \cup D'} = L_D \iota_{D'} + (-1)^{|D|} \iota_D L_D$, $[L_D, \iota_{D'}] = \iota_{[D,D']}$

where $D \cup D' = D(a_1, \cdots, a_m) D'(a_{m+1}, \cdots, a_{m+n})$

$\dashrightarrow$ homotopy Gerstenhaber alg str. on $C^*(A,A)$ &

G_∞-mod str on $C_*(A)$.

Cartan homotopy formula: homotopy Batallin-Vilkovitsky versions.