K (alg clos) field char 0.

X smooth var dim d over K

$X = \bigcup_\alpha U_\alpha$ $\quad \iota_\alpha : U_\alpha \hookrightarrow K^d$ as Zariski opn.

$\iota_\beta \iota_\alpha^{-1} : \iota_\alpha(U_\alpha \cap U_\beta) \longrightarrow \iota_\beta(U_\alpha \cap U_\beta)$ reg.

⇝ tangent sheaf $(T_U)_{U \subseteq_{opn} X}$.

polyvector fields sheaf $T_{poly} = (\oplus \Lambda^k T_U)_{U \subset X}$.

Čech cohomology $H^\bullet(X; T_{poly})$

Gerstenhaber algebra structure on $H^\bullet(X; T_{poly})$ :

1. gr. comm. prod : ext prod $X . Y = X \wedge Y$.
2. gr. Lie alg (shited by deg 1) : $[X, YZ] = [X,Y]Z + (-1)^{\cdot} Y[X$

Schouten-Nijenhuis bracket (⇝ Lie brack of v. fds.)

Goal: construct G-a'g derivations on $H^\bullet(X; T_{poly})$
( deriv for both $\Lambda$Lie-a'g & Com-a'g str. )

Ingredients 0. Deformation complex.

1. Kontsevich's graph complex(es) $Gra(n)$, $fGC$, $GC$.

$GC \subseteq_{qis} fGC \longrightarrow Def_{\Lambda Lie}(T_{poly})$ when $X = K^d$ or infin neigh of $0 \in K^d$.

2. Fedosov resolution $(T_{poly\,U} \longrightarrow F_U)_{U \subset_{op} X}$

Chern forms $c_k \in \check{C}^k((U_\alpha)_\alpha ; \Lambda^k T^* \otimes \mathfrak{gl}_d)$

$(\rightsquigarrow H^{k,k}(X) \subset H^{2k}(X))$.

3. Globalizing the graph action : $GC \to Def_{\Lambda Lie}(F)$.

- compatibility w/ Com-a'g str.

scheme : $H^\bullet(X; T_{poly}) \underset{\text{step 2.}}{\xrightarrow{\simeq}} H^\bullet(X; F) \circlearrowleft H^\bullet(GC)$ step 1 & $c_1$.

- action of 

$\equiv$ contraction w/ $c_k$

$\in$ $Z^\bullet(GC)$

0. Deformation complex $(V, \partial)$ cplx ($\partial = 0$ except for ...)

dg Lie alg structure on $V$ $\equiv$ (shifted) (conilp) coderiv. $Q$ on the "cofree" cocom coalg $\overset{\sim}{\Sigma}$ $\mathrm{coCom}(V)$

s.t. $[\partial, Q] + \frac{1}{2}[Q, Q] = 0$.
$(Q \circ Q.)$.

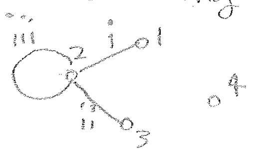
Deformation cplx : $\mathrm{Def}_{Lie}(V) = (\underline{\mathrm{coker}(\Sigma\, \mathrm{coCom}(V))}, [\partial + Q, -])$.
$\prod_n (s^{-n-1}(V^{\otimes n})^{S_n} \to V)$.

Prop. $Q' \in Z^0(\mathrm{Def}_{Lie}(V)) \rightsquigarrow Q'_1$ def's a Lie alg derivation of the Lie alg $H^0(V; \partial)$.

(operadic reform : $\mathrm{Cobar}(S\, \mathrm{coCom}) \to_{qis} \mathrm{Lie}$.
$\rightsquigarrow$ $\mathrm{Def}_{Lie} = \mathrm{Conv}(S\, \mathrm{coCom}, \mathrm{Lie})$.
for Gerstenhaber alg, $\mathrm{Ger} = \mathrm{Com} \circ \mathcal{S}\, \mathrm{Lie}$.
$\rightsquigarrow$ $\mathrm{coCom} \circ \mathcal{S}\, \mathrm{coLie}$. does similar job.)

1. Kontsevich's graph cplxs.

$n \in \mathbb{N}$. $\mathrm{Gra}(n)$ : gr. vec. sp. gen'ed by :
deg $e$ : graphs w/ $n$-verts & $e$-edges.
ordering on verts, edges.

ex. 

[ $S_n \curvearrowright \mathrm{Gra}(n)$ perm. v. lab.
rel. $\Gamma^\sigma = \mathrm{sgn}(\sigma)\Gamma$.

$A = K[[t^1, \cdots, t^d]]$ formal neigh. of $0 \in K^d$.
action of $\Gamma \in \mathrm{Gra}(n)$ on $T_{poly} S_p(A) = A[\xi_1 \cdots \xi_d]$
deg 1, anti com.

$$\Gamma(v_1 \otimes \cdots \otimes v_n) = m^{(n-1)} \prod_{\substack{e \in E(\Gamma) \\ a \overset{e}{-} b}} \sum_{k=1}^{d} \left( \underset{a}{\partial_{t^k}} \otimes \underset{b}{\partial_{\xi_k}} + \underset{a}{\partial_{\xi_k}} \otimes \underset{b}{\partial_{t^k}} \right)$$

$$T_{poly} S_p(A)^{\otimes n} \longrightarrow T_{poly} S_p(A).$$

Example $\overset{1}{\circ}\ \overset{2}{\circ}$ $\rightsquigarrow$ prod. $\overset{1}{\circ}\!\overset{i}{-}\!\overset{2}{\circ}$ $\rightsquigarrow$ Lie bracket.

prod. str. of graphs : make the action mult. into coderivations.

$\Gamma \in \mathrm{Gra}(m)$, $\Gamma' \in \mathrm{Gra}(n)$ $\rightsquigarrow$ $\Gamma\Gamma' = \sum_{i=1}^{m} \Gamma \circ_i \Gamma' \in \mathrm{Gra}(m+n-1)$

$\Gamma \circ_i \Gamma'$ : repl. $\circ^i \in \Gamma$ by $\Gamma'$. reconn. edges to $\circ^i$ in all possible ways.

$\rightsquigarrow$ $\Big( \underbrace{\prod_n s^{2n-2}\, \mathrm{Gra}(n)}_{\| \atop \mathrm{fGC}},\ [\circ\!-\!\circ, -] \Big) \to \mathrm{Def}_{s\mathrm{Lie}}(T_{poly} S_p(A))$

2. Fedosov resolution.

$U_\alpha \subset X$ opn w/ global coord $x^1, \dots, x^d$.

for $f \in \mathcal{O}_X$ (or $\mathcal{O}_U$) and $\underline{i} \in \mathbb{N}^d$,

formally consider $f_{\underline{i}}$ s.t.

$f \longmapsto \tilde{f} = \sum f_{\underline{i}} t^{\underline{i}} \in \mathcal{O}_X[[t^1, \dots, t^d]]$

becomes an alg hom. "function on jets of $\{$loc. diffeo $K^d \to X$, $0 \mapsto x\}$.

$\mathcal{O}^{coord}_{U_\alpha} := \mathcal{O}_U[x^a_{\underline{i}} : a = 1, \dots, d,\ \underline{i} \in \mathbb{N}^d_{>0}] \,/\, \det(x^a_{(0,\dots,1,\dots,0)})$

Def. $T^p_{U_\alpha} = \Lambda^p T_{U_\alpha} \xrightarrow{\tau} \mathcal{O}^{coord}_{U_\alpha} \otimes T^p S_p(A)$

$$f\, \partial_{x^{a_1}} \wedge \cdots \wedge \partial_{x^{a_p}} \longmapsto \sum_{b} \tilde{f} \prod J^{-1}{}_{a_i, b_i}\, \partial_{t^{b_1}} \wedge \cdots \wedge \partial_{t^{b_p}}.$$

$$J_{a,b} = \partial_{t^b} \widetilde{x^a}.$$

Prop. $\mathfrak{g}\mathrm{ld}$ acts on $\mathcal{O}^{cood}_{U_\alpha} \otimes T^p S_p(A)$. Image of $\tau$ is invariant under $\mathfrak{g}\mathrm{ld}$.

($\tau$ = natural lifting of Jets $\to X$.)

The 1-form $\omega$ : on $U_\alpha$,

$$\omega^\alpha = - J^{-1}_{a,b} \sum_{\underline{i}} dx^b_{\underline{i}}\, t^{\underline{i}} \otimes \partial_{t^a}.$$

Rem 1. $\omega^\alpha \tilde{x}^a + d\hat{x}^a = 0$ in $\Omega^1_K(\mathcal{O}^{coord}_{U_\alpha})[[t^\bullet]]$

$\rightsquigarrow \omega^\alpha \tilde{f} + d\hat{f} = 0 \quad \forall f$.

2. $(\omega^\alpha)_\alpha$ patch together $\rightsquigarrow \in \Omega^1_K(\mathcal{O}^{coord}_X)[[t]]$

Fedosov res $F^\bullet = \left( \left( \Omega^\bullet_K(\mathcal{O}^{coord}) \otimes T^\bullet_{poly} Sp(A) \right)^{gl_d}, d + [\omega, -] \right)$

Thm $\mathcal{O}_{U_\alpha} \xrightarrow{\tau} F^\bullet_{U_\alpha}$ is a resolution.

$F^\bullet$ is a dg Gerstenhaber algebra

differential $d + [\omega, -]$

bracket $[\xi \otimes v, \eta \otimes w] = \xi \wedge \eta \otimes [v, w]$.

Rem $\partial_{t^{b_1}} \partial_{t^{b_2}} \omega^\alpha \cdot dt^{b_1} \cdot dt^{b_2} \in \Omega^1_K(\mathcal{O}^{coord}_{U_\alpha}) \otimes T^{1,2} Sp(A)$

$\updownarrow$

Chern form $c_1 \in \check{C}^1((U_\alpha)_\alpha; T^\vee \otimes gl_d)$

3. Globalizing graph action.

define only $GC \subset_{qis} fGC = \prod s^{2n-2} Gra(n)$. graph s.t.

1) any vertex has valency $> 2$.

2) remains conn after removing any edge

$Q$: SLie-alg str. on $T_{poly} Sp(A)$

$\rightsquigarrow$ $Def_{SLie}(F^\bullet)$: cplx $d_{\Omega^\bullet_K(\mathcal{O}^{coord})} + [\omega, -]_{SN} + Q$

"$\Omega^\bullet_K(\mathcal{O}^{coord})$-linear" ext. of the graph action

$fGC \to (Def_{SLie}(F^\bullet), d + Q)$.

1) compose w/ $(Def_{SLie}(F^\bullet), d+Q) \xrightarrow{\sim} (Def_{SLie}(F^\bullet), d + [\omega, -]_{SN})$

2) $gl_d$-invariance.

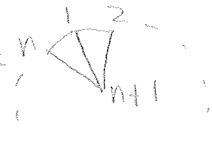
sol. 1) compose w/ $e^{\omega}$. ($[\omega, \cdot]_{SN} : [d, \text{mult. } \omega]$.)

2). only for $GC \subset fGC$.

point : $\iota_B \, \Gamma(\omega, \cdots \omega, v_1, v_2, \cdots)$

$$= \sum \Gamma(\omega, \cdots, B_{a,b} t^b \partial_{t^a}, \omega, \cdots, v_1, \cdots)$$

and valency cond.

Action of $\Gamma_n :=$  ; on $H^\cdot(X; T_{poly}) : H^\cdot(X; \mathcal{F}^\cdot)$

Need to compute $\Gamma_n(\omega, \cdots \omega, v)$ $\quad v \in \check{Z}^\cdot((U_\alpha)_k; \mathcal{F}^\cdot)$

$\omega \in \Omega^1(\mathcal{O}^{coord}) \otimes T\, S_p(A)$.

$\rightsquigarrow$ "$\partial_\xi$" can hit $\omega$ at most once.

$$\rightsquigarrow \Gamma_n(\omega, \cdots, \omega, v) = \Big( \prod_{a_i, b_i, c} \partial_{t^{a_i}} \partial_{t^{b_i}} \partial_{\xi^{b_i}} \omega \cdot d\tilde{x}^{a_i}, v \Big)$$

$$= (c_n, v).$$

- compatibility w/ (gr) comm alg str.